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Calculation of the convexity adjustment to the forward rate in the Vasicek model for the forward exotic contracts

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Abstract. In the following article, we consider forward contracts, which are financial instruments used to buy or sell some assets at a certain point in the future, and at the fixed price. Such contracts are customizable and traded over-the-counter, unlike futures, which are standardized contracts traded at exchanges. Particularly, we focus on in-arrears interest rate forward contracts (in-arrears FRA). The difference from the vanilla FRA: floating rate is immediately paid after it is fixed. We calculate the convexity adjustment to the forward simple interest rate in the single-factor Vasicek stochastic model for such contracts with different payment dates. With the help of the no-arbitrage market condition it is shown that such adjustments should be non-negative when payments occur before the end of accrual period and should be negative when payments occur after accrual period. We also studied in-arrears forward and option contracts, where fixed interest rate and principal, on which this rate is accrued, are denominated in different currencies (so called quanto in-arrears FRA and quanto in-arrears options). We checked that quanto in-arrears FRA equals in-arrears FRA in case when rates and principal are from the same currency market, and that quanto in-arrears option contract prices are greater than those of vanilla options.

Keywords: convexity adjustment; forward rate agreement (FRA); Vasicek model; no-arbitrage market; in-arrears FRA (iFRA) / quanto iFRA (quanto in-arrears FRA); quanto FRA; LIBOR; MOSPRIME.

JEL Classification: G12, G13, C02.

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1. INTRODUCTION

Forward contracts are widely used financial instruments used for purchase/sell of some asset at a certain date in future at the specified fixed price.

An example of forward contract is a forward rate agreement (FRA) on interest rate as an underlying asset, which we define in the next section.

FRA is a cash settled contract with the payment based on the net difference between the floating interest rate and the fixed rate (Hull, 2017). Fixed rate makes the initial price of the FRA being equal to 0 is called *forward rate*.

There is an exotic *in-arrears contract* which is settled at the beginning of the forward period — not at the end. The forward rate of an in-arrears contract is greater than the forward rate of a vanilla contract and the difference between these two rates depends on stochastic model used to simulate financial processes and is called convexity adjustment.

Studies on this topic may be found in (Mcinerney, Zastawniak, 2015), where LIBOR in-arrears rate was considered. The adjustment was calculated using the replication strategy and solving stochastic differential equation in the LIBOR market model. Another approach using the change of measure was studied in (Palsser, 2003), where simple lognormal stochastic model was chosen to calculate an in-arrears forward LIBOR rate. In (Gaminha, Gaspar, Oliveira, 2015), authors explored the Vasicek and Cox–Ingersoll–Ross models within LIBOR in-arrears rate. The authors obtained the adjustment from stochastic differential equation (SDE) numerical solution of convexity term SDE and found the partial closed-form solution for Vasicek model. There are also researches on in-arrears options — caps and floors (Hagan, 2003) where prices of options were found using the replication strategy for option-like pay-off. Finally, in the previous paper, written by two authors of this article, (Malykh, Postevoy, 2019), pricing of in-arrears FRA and in-arrears interest rate options using change of measure were considered.

There is also another kind of exotic forward contracts — quanto FRA, in which the notional principal amount is denominated in a currency other than the currency in which the interest rate is settled.

Such contracts were studied in (Lin, 2012), where author used forward measure pricing methodology to derive the valuation formulas within the Heath–Jarrow–Morton interest rate model. Research on quanto interest rate options may be found in (Hsieh, Chou, Chen, 2015), where authors also adopted martingale probability measure to obtain options pricing in the cross-currency LIBOR market model.

In this article we are going to continue our previous work and expand *change of measure method* in a single-factor Vasicek stochastic model (Vasicek, 1977) to consider cases, when the payment in FRA occurs in other dates,— not only at the beginning or at the end of the forward period. We prove that the convexity adjustment is negative when the settlement date takes place after the forward period. We also apply it to explore quanto FRA. Moreover, we combine it with the in-arrears FRA and come to the in-arrears quanto FRA. At the end, in-arrears quanto options are briefly considered.

2. DEFINITIONS

Let us introduce definitions which we use further in this paper.

Definition 1. Zero-coupon bond (ZCB) with maturity T is a security which promises to pay owner 1currency unit at T. We denote ZCB price at the moment t by $P(t,T)$, where $P(t,T)$ is an $\mathcal{F}_t$-measurable function and $P(T,T)=1$.

LIBOR is the indicative rate on which banks are willing to lend money each other, LIBID is the indicative rate on which banks are willing to borrow money. We assume equivalence of LIBID and LIBOR. MOSPRIME is a Russian analogue of the LIBOR rate, i.e. MOSPRIME is the indicative rate on which banks are willing to lend money to each other in rubles. We also make standard "Black–Sholes–Merton model" assumptions: no transaction costs; no default risk; no funding risk; no liquidity risk.

Now we define LIBOR rate and forward rate agreement more precisely.

Definition 2. We denote LIBOR spot rate at the moment t for a time period $\alpha>0$ by $L(t,t,t+\alpha)$. Bank can lend (or borrow) N currency units at the time t for a period α and get (return) $N(1+\alpha L(t,t,t+\alpha))$ currency units at the moment $t+\alpha$. Technically, MOPSRIME rate definition is similar to the LIBOR one, i.e. it is a spot rate with simple compounding. We use the LIBOR and MOSPRIME terms interchangeably through the article.

Definition 3. Forward rate agreement (FRA) is an over-the-counter contract for the exchange of two cash flows at a certain date in future. Floating reference rate is fixed at T_1. Buyer of this contract at $t\le T_1$ with maturity T_2, fixed rate K and principal N, agrees on following obligation between counterparties at T_2:

1) *pay* $(T_2-T_1)KN$ currency units to contract counterparty,

2) *receive* $(T_2-T_1)L(T_1,T_1,T_2)N$ currency units from contract counterparty.

The price of the FRA at T_2 is equal to $(T_2-T_1)(L(T_1,T_1,T_2)-K)N$.

For simplicity, we assume that principal amount $N=1$.

Definition 4. Forward rate $L(t,T_1,T_2)$ is the fixed rate K which makes price of the FRA contract at t equal to 0 for $t\le T_1\le T_2$.

It can be shown (Hull, 2017), that $L(t,T_1,T_2)=\dfrac{P(t,T_1)-P(t,T_2)}{(T_2-T_1)P(t,T_2)}$.

Now, we consider exotic in-arrears FRA: this contract is settled at time T_1.

Definition 5. In-arrears FRA (iFRA) is an over-the-counter contract for the exchange of two cash flows at a certain date. Floating reference rate is fixed at T_1. Buyer of this contract at $t\le T_1$ with maturity T_1, fixed rate K and principal N, agrees on following obligation between counterparties at T_1 (not T_2):

1) *pay* $(T_2 - T_1)KN$ currency units to counterparty,

2) *receive* $(T_2 - T_1)L(T_1, T_1, T_2)N$ currency units from counterparty.

The price of the iFRA at T_1 is equal to $(T_2 - T_1)(L(T_1, T_1, T_2) - K)N$.

We denote K which gives iFRA a 0 (zero) price at t by $iL(t, T_1, T_2)$.

A portfolio of assets is called self-financed if its value changes only due to changes in the asset prices.

Definition 6. Self-financed potrfolio A is called an arbitrage portfolio on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ if its price (value) at the time t is $V^A(t) \le 0$ and $\exists T > t : \mathbb{P}(V^A(T) \ge 0) = 1$ and $\mathbb{P}(V^A(T) > 0) > 0$.

We use the assumption of absence of any arbitrage portfolio on the market.

3. IN-ARREARS FRA

It was shown (Malykh, Postevoy, 2019) the that convexity adjustment (CA) for in-arrears FRA under single-factor Vasicek model is:

$$CA(t, T_1, T_2) = \frac{1}{T_2 - T_1} \times \frac{P(t, T_1)}{P(t, T_2)} (\mathrm{e}^I - 1),$$

where

$$I = \frac{\sigma^2}{a^2}\left(\frac{1}{2a} - \frac{1}{2a}\mathrm{e}^{-2a(T_1 - t)} - \frac{1}{a}\mathrm{e}^{-a(T_2 - T_1)} + \frac{1}{2a}\mathrm{e}^{-2a(T_2 - T_1)} + \frac{1}{a}\mathrm{e}^{-a(T_1 + T_2) + 2at} - \frac{1}{2a}\mathrm{e}^{-2a(T_2 - t)}\right),$$

$$P(t, T) = A(t, T)\mathrm{e}^{-B(t, T)r(t)},$$

$$B(t, T) = \left(1 - \mathrm{e}^{-a(T - t)}\right) / a,$$

$$A(t, T) = \exp\left(\left(B(t, T) - (T - t)\right)\left(\frac{\theta}{a} - \frac{\sigma^2}{2a^2}\right) - \frac{\sigma^2 B(t, T)^2}{4a}\right),$$

where θ and a are constant parameters in this model, which is given by the following SDE for instantaneous interest spot-rate: $dr(t) = (\theta - ar(t))dt + \sigma dW(t)$. Now we are going to study other exotic FRAs in this model.

4. EXOTIC FRA WITH DIFFERENT PAYMENT TIME OPTIONS

Along with the in-arrears contracts we can construct a FRA with payment date T_{pay} such as $t < T_{pay} < T_1$, $T_1 < T_{pay} < T_2$, or $T_2 < T_{pay}$. We consider each of these contracts using the same change of measure technique described in (Geman, Karoui, Rochet, 1995).

Let us denote exotic forward LIBOR rate by iL. Forward rate is the expected value of the future rate under appropriate forward measure (Privault, 2012). Then

$$iL(t, T_{pay}, T_1, T_2) = \mathrm{E}_{Q_{T_{pay}}}\left[L(T_1, T_1, T_2) \,|\, \mathcal{F}_t\right] \quad (1)$$

($E_{Q_{T_{pay}}}$ — conditional expectation value).

Theorem 1. *In a single-factor Vasicek model we have*

$$iL(t, T_{pay}, T_1, T_2) = L(t, T_1, T_2) + L(t, T_1, T_2)\frac{P(t, T_2)1_{T_{pay} \le T_2} - P(t, T_{pay})}{P(t, T_{pay})} +$$
$$+ \frac{P(t, T_2)}{P(t, T_{pay})\tau}\left(\frac{P(t, T_1)}{P(t, T_2)}\frac{P(t, T_{pay})}{P(t, T_2)}\mathrm{e}^I - \left(\frac{P(t, T_1)}{P(t, T_2)} - 1\right)1_{T_{pay} \le T_2} - \frac{P(t, T_{pay})}{P(t, T_2)}\right), \quad (2)$$

where

$$I = \frac{\sigma}{2a^3}\Big(\mathrm{e}^{-a|T_1 - T_{pay}|} - \mathrm{e}^{-a(T_1 + T_{pay} - 2t)} - \mathrm{e}^{-a(T_2 - \min(T_1, T_{pay}))} + \mathrm{e}^{-a(T_2 - 2t + \min(T_1, T_{pay}))} - \mathrm{e}^{-a(T_2 + \max(T_1 - 2T_{pay}, T_{pay} - 2T_1))} +$$
$$+ \mathrm{e}^{-a(T_2 - 2t + \max(T_1, T_{pay}))} + \mathrm{e}^{-2a(T_2 - \min(T_1, T_{pay}))} - \mathrm{e}^{-2a(T_2 - t)}\Big)$$

($1_{T_{pay} \le T_2}$ — *indicator function*).

The case $T_{pay} = T_1$ is considered in (Malykh, Postevoy, 2019). Now we consider other cases.

4.1. $t < T_{pay} < T_1$

Using results from (Privault, 2012), we change the measure to Q_{T_2} in (1).

$$iL(t,T_{pay},T_1,T_2)=\frac{P(t,T_2)}{P(t,T_{pay})}\mathrm{E}_{Q_{T_2}}\left[L(T_1,T_1,T_2)\frac{1}{P(T_{pay},T_2)}|\mathcal{F}_t\right]=$$

$$=\frac{P(t,T_2)}{P(t,T_{pay})}\mathrm{E}_{Q_{T_2}}\left[L(T_1,T_1,T_2)\big(1+(T_2-T_{pay})L(T_{pay},T_{pay},T_2)\big)|\mathcal{F}_t\right]=$$

$$=\frac{P(t,T_2)}{P(t,T_{pay})}\Big(L(t,T_1,T_2)+(T_2-T_{pay})\mathrm{E}_{Q_{T_2}}\left[L(T_1,T_1,T_2)L(T_{pay},T_{pay},T_2)|\mathcal{F}_t\right]\Big).$$

Using the tower property of conditional expectation:

$$\mathrm{E}_{Q_{T_2}}\left[L(T_1,T_1,T_2)L(T_{pay},T_{pay},T_2)|\mathcal{F}_t\right]=\mathrm{E}_{Q_{T_2}}\left[\mathrm{E}_{Q_{T_2}}\left[L(T_1,T_1,T_2)L(T_{pay},T_{pay},T_2)|\mathcal{F}_{T_{pay}}\right]|\mathcal{F}_t\right]=$$

$$=\mathrm{E}_{Q_{T_2}}\left[L(T_{pay},T_1,T_2)L(T_{pay},T_{pay},T_2)|\mathcal{F}_t\right].$$

Next we find dynamic of the following process under Q_{T_2}-measure:

$$L(T_{pay},T_1,T_2)L(T_{pay},T_{pay},T_2)=$$

$$=\frac{1}{(T_2-T_1)(T_2-T_{pay})}\left(\frac{P(T_{pay},T_1)P(T_{pay},T_{pay})}{P(T_{pay},T_2)P(T_{pay},T_2)}-\frac{P(T_{pay},T_1)}{P(T_{pay},T_2)}-\frac{P(T_{pay},T_{pay})}{P(T_{pay},T_2)}+1\right).$$

The 2nd and the 3rd terms are the martingales under Q_{T_2}-measure. We need to know dynamic of the 1st term.

$$d\left(\frac{P(t,T_1)P(t,T_{pay})}{P(t,T_2)P(t,T_2)}\right)=$$

$$=\frac{P(t,T_1)P(t,T_{pay})}{P(t,T_2)P(t,T_2)}\Big((\zeta^{T_{pay}}(t)+\zeta^{T_1}(t)-2\zeta^{T_2}(t))dW^{T_2}(t)+(\zeta^{T_{pay}}(t)-\zeta^{T_2}(t))(\zeta^{T_1}(t)-\zeta^{T_2}(t))dt\Big),$$

where $\zeta^{T_i}(t)=\sigma B(t,T_i)$. So,

$$\frac{P(T_{pay},T_1)P(T_{pay},T_{pay})}{P(T_{pay},T_2)P(T_{pay},T_2)}=\frac{P(t,T_1)P(t,T_{pay})}{P(t,T_2)P(t,T_2)}-\exp\Big[\int_t^{T_{pay}}\Big((\zeta^{T_{pay}}(t)+\zeta^{T_1}(t)-2\zeta^{T_2}(t)\Big)dW^{T_2}(t)+$$

$$+\Big(\zeta^{T_{pay}}(t)-\zeta^{T_2}(t)\Big)\Big(\zeta^{T_1}(t)-\zeta^{T_2}(t)\Big)-0.5\Big(\zeta^{T_{pay}}(t)+\zeta^{T_1}(t)-2\zeta^{T_2}(t))^2dt\Big)\Big].$$

Now we find expectation under Q_{T_2}-measure:

$$\mathrm{E}_{Q_{T_2}}\left[\frac{P(T_{pay},T_1)P(T_{pay},T_{pay})}{P(T_{pay},T_2)P(T_{pay},T_2)}|\mathcal{F}_t\right]=\frac{P(t,T_1)P(t,T_{pay})}{P(t,T_2)P(t,T_2)}\mathrm{e}^I,$$

where

$$I=\int_t^{T_{pay}}(\zeta^{T_{pay}}(t)-\zeta^{T_2}(t))(\zeta^{T_1}(t)-\zeta^{T_2}(t))dt=\frac{\sigma^2}{2a^3}\Big(\mathrm{e}^{-a(T_1-T_{pay})}-\mathrm{e}^{-a(T_1+T_{pay}-2t)}-$$

$$-\mathrm{e}^{-a(T_2-T_{pay})}+\mathrm{e}^{-a(T_2+T_{pay}-2t)}-\mathrm{e}^{-a(T_1+T_2-2T_{pay})}+\mathrm{e}^{-a(T_1+T_2-2t)}+\mathrm{e}^{-2a(T_2-T_{pay})}-\mathrm{e}^{-2a(T_2-t)}\Big).$$

Putting it all together we can write

$$iL(t,T_{pay},T_1,T_2)=L(t,T_1,T_2)+L(t,T_1,T_2)P(t,T_2)-P(t,T_{pay})/P(t,T_{pay})+$$

$$+\frac{P(t,T_2)}{P(t,T_{pay})(T_2-T_1)}\left(\frac{P(t,T_1)P(t,T_{pay})}{P(t,T_2)P(t,T_2)}\mathrm{e}^I-\frac{P(t,T_1)}{P(t,T_2)}-\frac{P(t,T_{pay})}{P(t,T_2)}+1\right).$$

4.2. $T_1 < T_{pay} < T_2$

Under Q_{T_2}-measure:

$$iL(t,T_{pay},T_1,T_2)=\frac{P(t,T_2)}{P(t,T_{pay})}E_{Q_{T_2}}\left[L(T_1,T_1,T_2)\frac{1}{P(T_{pay},T_2)}|\mathcal{F}_t\right]=$$
$$=\frac{P(t,T_2)}{P(t,T_{pay})}E_{Q_{T_2}}\left[L(T_1,T_1,T_2)\left(1+(T_2-T_{pay})L(T_{pay},T_{pay},T_2)\right)|\mathcal{F}_t\right]=$$
$$=\frac{P(t,T_2)}{P(t,T_{pay})}\left(L(t,T_1,T_2)+(T_2-T_{pay})E_{Q_{T_2}}\left[L(T_1,T_1,T_2)L(T_{pay},T_{pay},T_2)|\mathcal{F}_t\right]\right)=$$
$$=\frac{P(t,T_2)}{P(t,T_{pay})}\left(L(t,T_1,T_2)+(T_2-T_{pay})E_{Q_{T_2}}\left[L(T_1,T_1,T_2)L(T_1,T_{pay},T_2)|\mathcal{F}_t\right]\right).$$

Using the same technique as in Section 4.1, we can find the solution for this contract:

$$iL(t,T_{pay},T_1,T_2)=L(t,T_1,T_2)+L(t,T_1,T_2)\left(P(t,T_2)-P(t,T_{pay})\right)/P(t,T_{pay})+$$
$$+\frac{P(t,T_2)}{P(t,T_{pay})(T_2-T_1)}\left(\frac{P(t,T_1)P(t,T_{pay})}{P(t,T_2)P(t,T_2)}e^{I}-\frac{P(t,T_1)}{P(t,T_2)}-\frac{P(t,T_{pay})}{P(t,T_2)}+1\right),$$

where

$$I=\int_t^{T_1}\left((\zeta^{T_{pay}}(t)-\zeta^{T_2}(t))(\zeta^{T_1}(t)-\zeta^{T_2}(t))\right)dt=\frac{\sigma^2}{2a^3}\left(e^{-a(T_{pay}-T_1)}-e^{-a(T_1+T_{pay}-2t)}-\right.$$
$$\left.-e^{-a(T_2-T_1)}+e^{-a(T_1+T_2-2t)}-e^{-a(T_2+T_{pay}-2T_1)}+e^{-a(T_2+T_{pay}-2t)}+e^{-2a(T_2-T_1)}-e^{-2a(T_2-t)}\right). \quad (3)$$

4.3. $T_2 < T_{pay}$

Forward LIBOR rate has the following formula in this time payment case:

$$iL(t,T_{pay},T_1,T_2)=\frac{P(t,T_2)}{P(t,T_{pay})(T_2-T_1)}\left(\frac{P(t,T_{pay})P(t,T_1)}{P(t,T_2)P(t,T_2)}e^{I}-\frac{P(t,T_{pay})}{P(t,T_2)}\right),$$

where I is taken from (3), as both cases take place after T_1.

In the case when payment occurs after accrual period, we can prove that adjustment should be always non-positive similarly to what we did in (Malykh, Postevoy, 2019) for payments before the end of accrual period.

Theorem 2. *Suppose that* $\mathbb{P}(L(T_1,T_1,T_2)\neq L(t,T_1,T_2))>0$ *under real-word measure. Then the forward rate* $iL(t,T_{pay},T_1,T_2)<$ *forward rate* $L(t,T_1,T_2)$, $t<T_1\le T_2<T_{pay}$.

P r o o f. We can prove it by contradiction assuming opposite and constructing an arbitrage portfolio.

Assume that there is a forward rate on the market and $iL(t,T_{pay},T_1,T_2)\ge L(t,T_1,T_2)$. Without loss of generality let $(T_2-T_1)=1$ year. Without loss of generality let $(T_2-T_1)=1$ year. Consider the following strategy:

1) time t: buy FRA with $K=L(t,T_1,T_2)$, $N=1$ and sell iFRA with payment date T_{pay}, $K=iL(t,T_{pay},T_1,T_2)$ and $N=P(t,T_1)/P(t,T_2)$. Portfolio value $V_t=0$;

2) T_1: LIBOR rate is fixed and we enter into forward contract to buy $(L(T_1,T_1,T_2)-L(t,T_1,T_2))P(T_1,T_2)/P(T_1,T_{pay})$ number of zero-coupon bonds (ZCB) with maturity T_{pay} at time T_2. It costs us $F=L(T_1,T_1,T_2)-L(t,T_1,T_2)$;

3) T_2: FRA settlement occurs. Portfolio value is
$$V_{T_2}=\left(L(T_1,T_1,T_2)-L(t,T_1,T_2)\right)-F+\left(L(T_1,T_1,T_2)-L(t,T_1,T_2)P(T_2,T_{pay})P(T_1,T_2)/P(T_1,T_{pay})\right);$$

4) T_{pay}: iFRA settlement occurs

$$V_{T_{pay}}=\left(L(T_1,T_1,T_2)-L(t,T_1,T_2)\right)\frac{P(T_1,T_2)}{P(T_1,T_{pay})}+\frac{P(t,T_1)}{P(t,T_2)}\left(iL(t,T_1,T_2)-L(T_1,T_1,T_2)\right).$$

We use the fact that $(T_2 - T_1)L(t,T_1,T_2) = P(t,T_1)/P(t,T_2) - 1$ and that $P(t,T_1) \geq P(t,T_2)\ \forall t \leq T_1 \leq T_2$. Now we can rewrite out portfolio value:

$$V_{T_{pay}} \geq (L(T_1,T_1,T_2) - L(t,T_1,T_2))\left(\frac{P(T_1,T_2)}{P(T_1,T_{pay})} - \frac{P(t,T_1)}{P(t,T_2)}\right) = \left(\frac{P(T_1,T_1)}{P(T_1,T_2)} - \frac{P(t,T_1)}{P(t,T_2)}\right) \times \left(\frac{P(T_1,T_2)}{P(T_1,T_{pay})} - \frac{P(t,T_1)}{P(t,T_2)}\right) =$$

$$= \frac{(P(T_1,T_1)P(t,T_2) - P(T_1,T_2)P(t,T_1))(P(T_1,T_2)P(t,T_2) - P(T_1,T_{pay})P(t,T_1))}{P(T_1,T_2)P(t,T_2)P(T_1,T_{pay})P(t,T_2)} \geq$$

$$\geq \frac{P(T_1,T_2)P(T_1,T_{pay})(P(t,T_2) - P(t,T_1))^2}{P(T_1,T_2)P(t,T_2)P(T_1,T_{pay})P(t,T_2)} \geq 0.$$

It's worth noting that $\mathbb{P}(V_{T_{pay}} > 0) > 0$, because of our assumption, that $\mathbb{P}(L(T_1,T_1,T_2) \neq L(t,T_1,T_2)) > 0$.

We managed to construct an arbitrage portfolio which contradicts to our assumption of no-arbitrage. Hence, $iL(t,T_{pay},T_1,T_2) < L(t,T_1,T_2)$.

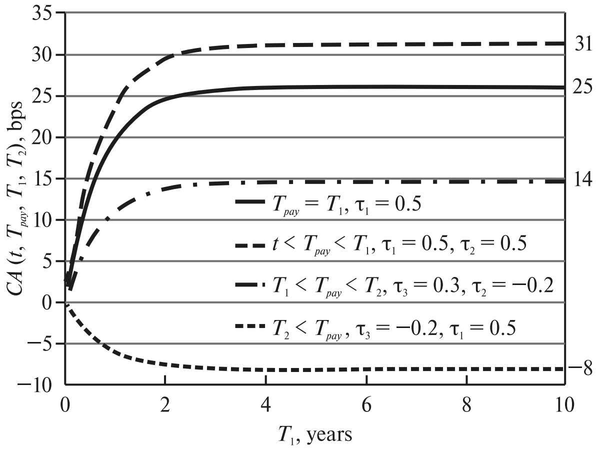

Fig. 1. Comparison of adjustments: *CA* (convexity adjustment) for forward LIBOR rate with $t = 0$; $\theta = 0.035$; $\tau = 0.5$; $r(t) = 5\%$ (bps — 1 basis point is equivalent to 0.01% (1/100th of a percent) or 0.0001 in decimal form)

5. IN-ARREARS FRA BEHAVIOR

Using results from Section 4.1–4.3 we proved the common formula (2). We can also find the limit of adjustments when $T_1 \to \infty$. We denote $\tau_1 = T_2 - T_1$, $\tau_2 = T_1 - T_{pay}$, $\tau_3 = T_2 - T_{pay}$. Then

$$\lim_{T_1 \to \infty} CA = \frac{1}{\tau_1} \exp\left\{\left(\frac{\theta}{a} - \frac{\sigma^2}{2a}\right)\tau_1\right\}(e^J - 1),$$

where

$$J = (\sigma^2 / 2a^3)(e^{-a|\tau_2|} - e^{-a((\tau_3)1_{\tau_2>0} + \tau_1 1_{\tau_2 \leq 0})} + e^{-2a((\tau_3)1_{\tau_2>0} + \tau_1 1_{\tau_2 \leq 0})} - e^{-a(|\tau_2| + \tau_3 + (-\tau_2)^+)}).$$

Using these properties convexity adjustment with different payment date properties is given in fig. 1.

6. QUANTO IN-ARREARS FRA

We consider another exotic modification of FRA — quanto FRA.

Definition 7. Quanto FRA is a forward contract, where buyer of this contract at $t \leq T_1$ with maturity T_2, fixed rate K in d-currency (domestic) units and principal N in f-currency (foreign) units, agrees on following obligations with counterparties at T_2:

1) byuer *pay* $(T_2 - T_1)KN$ f-currency units;

2) *receive* $(T_2 - T_1)L(T_1,T_1,T_2)N$ f-currency units, where L — LIBOR rate in d-currency units.

Definition 8. Quanto in-arrears FRA (iqFRA) is a forward contract, where buyer of this contract at $t \leq T_1$ with maturity T_2, fixed rate K in d-currency units and principal N in f-currency units, agrees on following obligations with counterparties at T_1:

1) byuer *pay* $(T_2 - T_1)KN$ f-currency units,

2) *receive* $(T_2 - T_1)L(T_1,T_1,T_2)N$ f-currency units, where L — LIBOR rate in d-currency units.

Let $N = 1$.

By *iqL* we denote forward rate of iqFRA contract. Notation $\mathbb{E}_{Q^f_{T_1}}$ means mathematical expectation by forward measure T_1 of payments in f-currency. Then $iqL(t,T_1,T_2) = \mathbb{E}_{Q^f_{T_1}}[L(T_1,T_1,T_2) | \mathcal{F}_t]$.

We need to change measure to $Q^d_{T_1}$ for payments in d-currency. Radon–Nikodym derivative is

$$\frac{dQ^d_{T_1}}{dQ^f_{T_1}} = \frac{P_d(T_1,T_1)}{P_d(t,T_1)} \frac{P_f(t,T_1)X(t)}{P_f(T_1,T_1)X(T_1)},$$

where $X(t)$ — spot exchange rate at time t. Then

$$iqL(t,T_1,T_2)=\mathbb{E}_{Q^d_{T_1}}\left[L(T_1,T_1,T_2)\frac{dQ^f_{T_1}}{dQ^d_{T_1}}|\mathcal{F}_t\right]=\frac{P_d(t,T_1)}{P_f(t,T_1)X(t)}\mathbb{E}_{Q^d_{T_1}}\left[L(T_1,T_1,T_2)\frac{P_f(T_1,T_1)}{P_d(T_1,T_1)}X(T_1)|\mathcal{F}_t\right]. \quad (4)$$

We use the fact that the forward exchange rate with maturity T is $X_T(t)=P_f(t,T)X(t)/P_d(t,T)$. Then $iqL(t,T_1,T_2)=\left(X_{T_1}(t)\right)^{-1}\mathbb{E}_{Q^d_{T_1}}\left[X_{T_1}(T_1)L(T_1,T_1,T_2)|\mathcal{F}_t\right]$.

To calculate this expectation we need to:

1) find SDE for process $X_{T_1}(t)$ in forward measure $Q^d_{T_1}$;

2) find joint distribution of $X_{T_1}(T_1)L(T_1,T_1,T_2)$ in forward measure $Q^d_{T_1}$.

First, write SDE of major processes:

$$\frac{dP_d(t,T)}{P_d(t,T)}=r_d(t)dt+\sigma_{P_d}B_{P_d}(t,T)dW^Q_{d,P_d}(t),\ \frac{dP_f(t,T)}{P_f(t,T)}=r_f(t)dt+\sigma_{P_f}B_{P_f}(t,T)dW^Q_{f,P_f}(t),$$

$$\frac{dX(t)}{X(t)}=(r_d(t)-r_f(t))dt+\sigma_X dW^Q_X(t).$$

W^Q_{d,P_d} means Wiener process for process P_d in measure Q in currency d. To find SDE of $X_{T_1}(t)$ in risk-neutral measure Q we need to write $P_f(t,T)$ in currency d. Changing the measure we get $\frac{dP_f(t,T)}{P_f(t,T)}=(r_f(t)-\zeta^T_{P_f}(t)\sigma_X\rho_{P_f,X})+\zeta^T_{P_f}(t)dW^Q_{d,P_f}(t)$, where $\zeta^T_{P_f}(t)=\sigma_{P_f}B_{P_f}(t,T)$ and $\rho_{P_f,X}$ — correlation between P_f and X. Now write SDE of $X_{T_1}(t)$:

$$d(X_{T_1})=\frac{\partial X_{T_1}}{\partial P_d}dP_d+\frac{\partial X_{T_1}}{\partial P_f}dP_f+\frac{\partial X_{T_1}}{\partial X}dX+0.5\frac{\partial^2 X_{T_1}}{\partial P_d^{\,2}}(dP_d)^2+\frac{\partial^2 X_{T_1}}{\partial P_d\partial P_f}(dP_d)(dP_f)+$$

$$+\frac{\partial^2 X_{T_1}}{\partial P_d\partial X}(dP_d)(dX)+\frac{\partial^2 X_{T_1}}{\partial P_f^2}(dP_f)^2.$$

Switching to Q^{T_1}-measure:

$$d(X_{T_1})/X_{T_1}=-\zeta^{T_1}_{P_d}(t)dW^{T_1}_{d,P_d}(t)+\zeta^{T_1}_{P_f}(t)dW^{T_1}_{d,P_f}(t)+\sigma_X dW^{T_1}_{d,X}(t),$$

$$d\ln(X_{T_1})=-\zeta^{T_1}_{P_d}(t)dW^{T_1}_{d,P_d}(t)+\zeta^{T_1}_{P_f}(t)dW^{T_1}_{d,P_f}(t)+\sigma_X dW^{T_1}_{d,X}(t)-$$

$$-0.5\left(-\zeta^{T_1}_{P_d}(t)dW^{T_1}_{d,P_d}(t)+\zeta^{T_1}_{P_f}(t)dW^{T_1}_{d,P_f}(t)+\sigma_X dW^{T_1}_{d,X}(t)\right)^2,$$

$$X_{T_1}(T_1)=X_{T_1}(t)\exp\left(\int_t^{T_1}\left(-\zeta^{T_1}_{P_d}(t)dW^{T_1}_{d,P_d}(t)+\zeta^{T_1}_{P_f}(t)dW^{T_1}_{d,P_f}(t)+\sigma_X dW^{T_1}_{d,X}(t)\right)-\right.$$

$$\left.-0.5\int_t^{T_1}\left(-\zeta^{T_1}_{P_d}(t)dW^{T_1}_{d,P_d}(t)+\zeta^{T_1}_{P_f}(t)dW^{T_1}_{d,P_f}(t)+\sigma_X dW^{T_1}_{d,X}(t)\right)^2\right).$$

Now we need to get $L(t,T_1,T_2)$ in Q^{T_1}-measure. Remember that $L(t,T_1,T_2)=(P_d(t,T_1)/P_d(t,T_2)-1)/(T_2-T_1)$. Then, recall that:

$$d\left(\frac{P_d(t,T_1)}{P_d(t,T_2)}\right)=\frac{P_d(t,T_1)}{P_d(t,T_2)}\left(\zeta^{T_1}_{P_d}(t)-\zeta^{T_2}_{P_d}(t)\right)\left(dW^Q_{d,P_d}(t)-\zeta^{T_2}_{P_d}(t)dt\right).$$

Changing measure to Q^{T_1}:

$$d\left(\frac{P_d(t,T_1)}{P_d(t,T_2)}\right)=\frac{P_d(t,T_1)}{P_d(t,T_2)}\left(\left(\zeta^{T_1}_{P_d}(t)-\zeta^{T_2}_{P_d}(t)\right)dW^{T_1}_{d,P_d}(t)+\left(\zeta^{T_1}_{P_d}(t)-\zeta^{T_2}_{P_d}(t)\right)^2dt\right),$$

$$d\ln\left(\frac{P_d(t,T_1)}{P_d(t,T_2)}\right)=\left(\zeta^{T_1}_{P_d}(t)-\zeta^{T_2}_{P_d}(t)\right)dW^{T_1}_{d,P_d}(t)+0.5\left(\zeta^{T_1}_{P_d}(t)-\zeta^{T_2}_{P_d}(t)\right)^2dt,$$

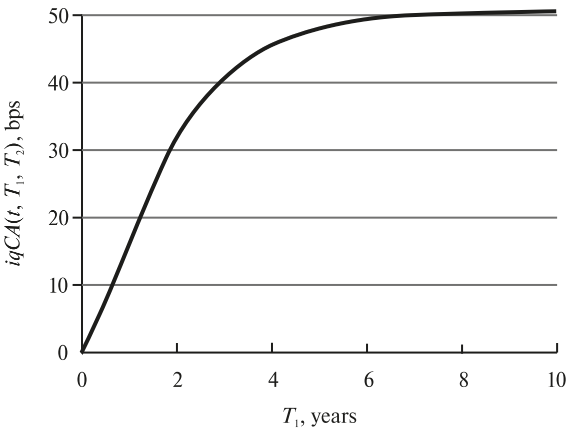

Fig. 2. Convexity adjustment for quanto in-arrears FRA with te following parameters: $t = 0$; $\sigma_d = \sigma_f = 10\%$; $T_2 - T_1 = 0.5$; $\theta_f = \theta_d = 0.035$; $r_d(f) = 5\%$; $r_f(f) = 10\%$; $\rho_{P_d,P_f} = \rho_{P_d,X} = \rho_{P_f,X} = 0.3$; $X(t) = 1$

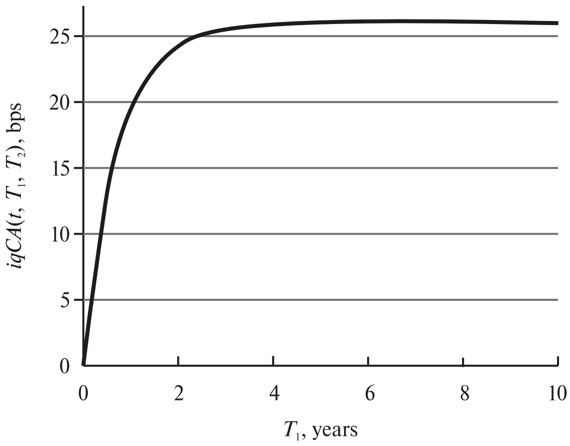

Fig. 3. Convexity adjustment for quanto in-arrears FRA with the following parameters: $t = 0$; $\sigma_d = \sigma_f = 10\%$; $\theta = 0.035$; $T_2 - T_1 = 0.5$; $a = 0.7$; $r(t) = 5\%$; $\rho_{P_d,X} = \rho_{P_f,X} = 0$; $\rho_{P_d,P_f} = 1$, both rates are identical

$$\frac{P_d(T_1,T_1)}{P_d(T_1,T_2)}=\frac{P_d(t,T_1)}{P_d(t,T_2)}\exp\left\{\frac{1}{2}\int_t^{T_1}\left(\zeta_{P_d}^{T_1}(t)-\zeta_{P_d}^{T_2}(t)\right)^2 dt+\int_t^{T_1}\left(\zeta_{P_d}^{T_1}(t)-\zeta_{P_d}^{T_2}(t)\right)dW_{d,P_d}^{T_1}(t)\right\}.$$

So,

$$X_{T_1}(T_1)\frac{P_d(T_1,T_1)}{P_d(T_1,T_2)}=X_{T_1}(t)\frac{P_d(t,T_1)}{P_d(t,T_2)}\exp\Big(0.5\int_t^{T_1}\Big(-2\zeta_{P_d}^{T_1}(t)\zeta_{P_d}^{T_2}(t)+\zeta_{P_d}^{T_2}(t)^2+2\zeta_{P_d}^{T_1}(t)\zeta_{P_f}^{T_1}(t)\rho_{P_d,P_f}-$$
$$-\zeta_{P_f}^{T_1}(t)^2+2\zeta_{P_d}^{T_1}(t)\sigma_X\rho_{P_d,X}-2\zeta_{P_f}^{T_1}(t)\sigma_X\rho_{P_f,X}-\sigma_X^2\Big)dt+\int_t^{T_1}-\zeta_{P_d}^{T_2}(t)dW_{d,P_d}^{T_1}(t)+$$
$$+\int_t^{T_1}\zeta_{P_f}^{T_1}(t)dW_{d,P_f}^{T_1}(t)+\int_t^{T_1}\sigma_X dW_{d,X}^{T_1}(t)\Big).$$

Using Cholesky decomposition, we decompose correlated wiener processes on independent ones:

$$dW_{d,P_d}^{T_1}(t)=a_{11}dB_1(t)+a_{12}dB_2(t)+a_{13}dB_3(t),\ dW_{d,P_f}^{T_1}(t)=a_{21}dB_1(t)+a_{22}dB_2(t)+a_{23}dB_3(t),$$
$$dW_{d,X}^{T_1}(t)=a_{31}dB_1(t)+a_{32}dB_2(t)+a_{33}dB_3(t),$$

where $dB_1(t)$, $dB_2(t)$, and $dB_3(t)$ — uncorrelated Wiener processes in Q^{T_1}-measure, a_{ij} are the elements of the covariance matrix square root. So, expectation of lognormal random variable:

$$\mathbb{E}_{Q_{T_1}^d}\left[X_{T_1}(T_1)\frac{P_d(T_1,T_1)}{P_d(T_1,T_2)}|\mathcal{F}_t\right]=X_{T_1}(t)\frac{P_d(t,T_1)}{P_d(t,T_2)}\mathrm{e}^I,$$

where

$$I=0.5\int_t^{T_1}(-2\zeta_{P_d}^{T_1}(t)\zeta_{P_d}^{T_2}(t)+\zeta_{P_d}^{T_2}(t)^2+2\zeta_{P_d}^{T_1}(t)\zeta_{P_f}^{T_1}(t)\rho_{P_d,P_f}-\zeta_{P_f}^{T_1}(t)^2+2\zeta_{P_d}^{T_1}(t)\sigma_X\rho_{P_d,X}-$$
$$-2\zeta_{P_f}^{T_1}(t)\sigma_X\rho_{P_f,X}-\sigma_X^2)dt+0.5\int_t^{T_1}(-\zeta_{P_d}^{T_2}(t)a_{11}+\zeta_{P_f}^{T_1}(t)a_{21}+\sigma_X a_{31})^2dt+$$
$$+0.5\int_t^{T_1}(-\zeta_{P_d}^{T_2}(t)a_{12}+\zeta_{P_f}^{T_1}(t)a_{22}+\sigma_X a_{32})^2dt+0.5\int_t^{T_1}(-\zeta_{P_d}^{T_2}(t)a_{13}+\zeta_{P_f}^{T_1}(t)a_{23}+\sigma_X a_{33})^2dt.$$

Calculating this expression, we get the equation for the in-arrears quanto FRA:

$$iqL(t,T_1,T_2)=\frac{1}{T_2-T_1}\left(\frac{P_d(t,T_1)}{P_d(t,T_2)}\mathrm{e}^I-\frac{1}{X_{T_1}(t)}\right),$$

where calculation of I is given in Appendix.

Convexity adjustment for this exotic forward contract is $iqCA(t,T_1,T_2)=iqL(t,T_1,T_2)-L(t,T_1,T_2)$. Fig. 2–3 show $iqCA(t,T_1,T_2)$ with different parameters.

In case when both rates are from the same currency market, adjustment term is similar to the in-arrears one, which is shown in the fig. 3.

7. QUANTO IN-ARREARS OPTIONS

As a part of our study of quanto in-arrears contracts we also consider quanto in-arrears options on interest rate — caplet and floorlet.

Definition 9. An in-arrears quanto caplet (floorlet) is a European-style call (put) option on interest rate which is fixed at T_1. Buyer of this option at $t \le T_1$ with maturity T_1, strike K and principal amount N is offered with the following rights at time T_1:

1) *pay (receive)* $(T_2 - T_1)KN$ f-currency units;

2) *receive (pay)* $(T_2 - T_1)L(T_1, T_1, T_2)N$ f-currency units, while L and K are set in d-currency units.

Formulas for option prices are given below:

$$qCpl(t,T_1,T_2,K) = (T_2 - T_1)P_f(t,T_1)E_{Q^f_{T_1}}\left[\left(L(T_1,T_1,T_2) - K\right)^+ \mid \mathcal{F}_t\right],$$

$$qFl(t,T_1,T_2,K) = (T_2 - T_1)P_f(t,T_1)E_{Q^f_{T_1}}\left[\left(K - L(T_1,T_1,T_2)\right)^+ \mid \mathcal{F}_t\right].$$

First, we find price of qCpl. We switch to d-currency — as in Section 6:

$$E_{Q^f_{T_1}}\left[(L(T_1,T_1,T_2) - K)^+ \mid \mathcal{F}_t\right] = E_{Q^d_{T_1}}\left[\left(L(T_1,T_1,T_2) - K\right)^+ \frac{P_f(T_1,T_1)}{P_d(T_1,T_1)} X(T_1) \mid \mathcal{F}_t\right] \frac{P_d(t,T_1)}{P_f(t,T_1)X(t)} =$$

$$= X^{-1}_{T_1}(t)E_{Q^d_{T_1}}\left[\left(L(T_1,T_1,T_2) - K\right)^+ X_{T_1}(T_1) \mid \mathcal{F}_t\right] = X^{-1}_{T_1}(t)E_{Q^d_{T_1}}\left[\frac{P_d(T_1,T_1)}{P_d(T_1,T_2)} X_{T_1}(T_1) \times 1_{\frac{P_d(T_1,T_1)}{P_d(T_1,T_2)} > 1 + (T_2 - T_1)K} \mid \mathcal{F}_t\right] -$$

$$- X^{-1}_{T_1}(t)E_{Q^d_{T_1}}\left[\left(1 + (T_2 - T_1)K\right) X_{T_1}(T_1) \times 1_{\frac{P_d(T_1,T_1)}{P_d(T_1,T_2)} > 1 + (T_2 - T_1)K} \mid \mathcal{F}_t\right].$$

Calculating both mathematical expectations, we come to the analytical formula of the quanto in-arrears option price:

$$qCpl(t,T_1,T_2,K) = P_f(t,T_1)\Big(\left\{P_d(t,T_1)/P_d(t,T_2)\right\}\exp\left\{0.5J_0\right\}\exp\left\{0.5(J_1 + J_2 + J_3)\right\} \times$$
$$\times N(\sqrt{J_1} - l)N(\sqrt{J_2} - l)N(\sqrt{J_3} - l) - \left(1 + (T_2 - T_1)K\right)\exp\left\{-0.5Q_0\right\} \times$$
$$\times \exp\left\{0.5(Q_1 + Q_2 + Q_3)\right\} N(\sqrt{Q_1} - l)N(\sqrt{Q_2} - l)N(\sqrt{Q_3} - l)\Big),$$

where

$$J_0 = \int_t^{T_1}(-2\zeta^{T_1}_{P_d}(t)\zeta^{T_2}_{P_d}(t) + \zeta^{T_2}_{P_d}(t)^2 + 2\zeta^{T_1}_{P_d}(t)\zeta^{T_1}_{P_f}(t)\rho_{P_d,P_f} - \zeta^{T_1}_{P_f}(t)^2 + 2\zeta^{T_1}_{P_d}(t)\sigma_X\rho_{P_d,X} - 2\zeta^{T_1}_{P_f}(t)\sigma_x\rho_{P_f,X} - \sigma^2_X)dt,$$

$$J_1 = \int_t^{T_1}(-\zeta^{T_2}_{P_d}(t)a_{11} + \zeta^{T_1}_{P_f}(t)a_{21} + \sigma_X a_{31})^2 dt,\ J_2 = \int_t^{T_1}(-\zeta^{T_2}_{P_d}(t)a_{12} + \zeta^{T_1}_{P_f}(t)a_{22} + \sigma_X a_{32})^2 dt,$$

$$J_3 = \int_t^{T_1}(-\zeta^{T_2}_{P_d}(t)a_{13} + \zeta^{T_1}_{P_f}(t)a_{23} + \sigma_X a_{33})^2 dt,$$

$$Q_0 = \int_t^{T_1}(\zeta^{T_1}_{P_d}(t)^2 - 2\zeta^{T_1}_{P_d}(t)\zeta^{T_1}_{P_f}(t)\rho_{P_d,P_f} - 2\zeta^{T_1}_{P_d}(t)\sigma_X\rho_{P_d,X} + \zeta^{T_1}_{P_f}(t)^2 + 2\zeta^{T_1}_{P_f}(t)\sigma_X\rho_{P_f,X} + \sigma^2_X)dt,$$

$$Q_1 = \int_t^{T_1}(-\zeta^{T_1}_{P_d}(t)a_{11} + \zeta^{T_1}_{P_f}(t)a_{21} + \sigma_X a_{31})^2 dt,\ Q_2 = \int_t^{T_1}(-\zeta^{T_1}_{P_d}(t)a_{12} + \zeta^{T_1}_{P_f}(t)a_{22} + \sigma_X a_{32})^2 dt,$$

$$Q_3 = \int_t^{T_1}(-\zeta^{T_1}_{P_d}(t)a_{13} + \zeta^{T_1}_{P_f}(t)a_{23} + \sigma_X a_{33})^2 dt,$$

calculations of which are given in Appendix.

We will find the floorlet price using put-call parity of European options:

$$qFl(t,T_1,T_2,K) = qCpl(t,T_1,T_2,K) - (T_2 - T_1)P_f(t,T_1)(iqL(t,T_1,T_2) - K).$$

Fig. 4–5 show differences in quanto in arrears and standard caplet and floorlet prices with different parameters.

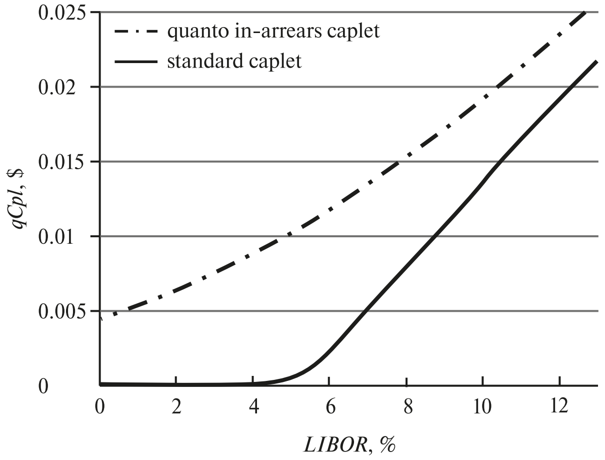

Fig. 4. Quanto in-arrears caplet price vs. Standard caplet price with the following parameters: $t=0$; $\sigma_d=\sigma_f=10\%$; $T_{i+1}-T_i=0.5$; $\theta_f=\theta_d=0.035$; $r_d(t)=5\%$; $r_f(t)=10\%$; $\rho_{Pd,Pf}=\rho_{Pd,X}=\rho_{Pf,X}=0.3$; $X(t)=1$

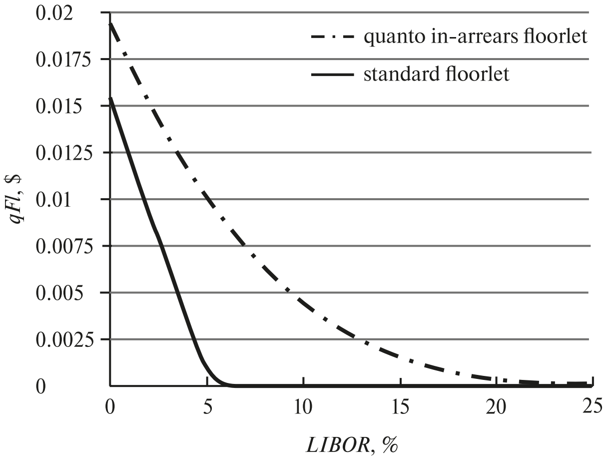

Fig. 5. Quanto in-arrears floorlet price vs. Standard floorlet price with the following parameters: $t=0$; $\sigma_d=\sigma_f=10\%$; $T_{i+1}-T_i=0.5$; $\theta_f=\theta_d=0.035$; $r_d(t)=5\%$; $r_f(t)=10\%$; $\rho_{Pd,Pf}=\rho_{Pd,X}=\rho_{Pf,X}=0.3$; $X(t)=1$

8. CONCLUSION

We derived the formula for calculating the forward LIBOR rate in FRA when payment is settled at different dates. It was proved that the convexity adjustment to the vanilla forward rate should be negative when payment takes place after forward period. Next, we studied quanto in-arrears FRA and checked, that it equals in-arrears FRA in case when rates and principal are from the same currency market, which is shown in the fig. 3. Finally, we briefly studied quanto in-arrears option contracts and found that their prices are greater than those of vanilla options.

APPENDIX

Here is the calculation of the integral from the Section 6: $I=I_0+\ldots+I_3$, where calculations of I_i, $i=0,\ldots,3$, are given below:

$$I_0=0.5\Bigg[\left(\sigma_{P_d}/a_{P_d}\right)^2\Bigg(T_1-t-\frac{\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_2-t)\right\}}{a_{P_d}}-\frac{1}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_1-t)\right\}}{a_{P_d}}+$$

$$+\frac{\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{2a_{P_d}}-\frac{\exp\left\{-a_{P_d}(T_2+T_1-2t)\right\}}{2a_{P_d}}\Bigg)+\left(\sigma_{P_d}/a_{P_d}\right)^2\Bigg(T_1-t-\frac{2\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{a_{P_d}}+$$

$$+\frac{2\exp\left\{-a_{P_d}(T_2-t)\right\}}{a_{P_d}}+\frac{\exp\left\{-2a_{P_d}(T_2-T_1)\right\}}{2a_{P_d}}-\frac{\exp\left\{-2a_{P_d}(T_2-t)\right\}}{2a_{P_d}}\Bigg)+$$

$$+2\rho_{P_d,P_f}\frac{\sigma_{P_d}\sigma_{P_f}}{a_{P_d}a_{P_f}}\Bigg(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}+\frac{1}{a_{P_d}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_d}}+\frac{1}{a_{P_d}+a_{P_f}}-$$

$$-\frac{\exp\left\{-\left(a_{P_d}+a_{P_f}\right)(T_1-t)\right\}}{a_{P_d}+a_{P_f}}\Bigg)-\left(\sigma_{P_f}/a_{P_f}\right)^2\Bigg(T_1-t-\frac{2}{a_{P_f}}+\frac{2\exp\left\{-a_{P_f}(T_2-t)\right\}}{a_{P_f}}+\frac{1}{2a_{P_f}}-$$

$$-\frac{\exp\left\{-2a_{P_f}(T_1-t)\right\}}{2a_{P_f}}\Bigg)+2\sigma_X\rho_{P_d,X}\frac{\sigma_{P_d}}{a_{P_d}}\left(T_1-t-\frac{1}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_1-t)\right\}}{a_{P_d}}\right)-$$

$$-2\sigma_X\rho_{P_f,X}\frac{\sigma_{P_f}}{a_{P_f}}\left(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}\right)-\sigma_X^2\left(T_1-t\right)\Bigg],$$

$$I_i=0.5\Bigg[\left(\sigma_{P_d}a_{1i}/a_{P_d}\right)^2\Bigg(T_1-t-\frac{\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{a_{P_d}}+\frac{2\exp\left\{-a_{P_d}(T_2-t)\right\}}{a_{P_d}}+\frac{\exp\left\{-2a_{P_d}(T_2-T_1)\right\}}{a_{P_d}}+$$

$$+\frac{\exp\left\{-2a_{P_d}(T_2-t)\right\}}{2a_{P_d}}\Bigg)-2a_{1i}a_{2i}\frac{\sigma_{P_d}\sigma_{P_f}}{a_{P_d}a_{P_f}}\Bigg(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}-\frac{\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{2a_{P_d}}+$$

$$+\frac{\exp\left\{-a_{P_d}(T_2-t)\right\}}{2a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{a_{P_d}+a_{P_f}}-\frac{\exp\left\{-2a_{P_d}(T_2-t)-a_{P_f}(T_1-t)\right\}}{a_{P_d}+a_{P_f}}\Bigg)-2a_{1i}a_{2i}\sigma_X\frac{\sigma_{P_d}}{a_{P_d}}\times$$

$$\times\left(T_1-t-\frac{\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_d-t)\right\}}{a_{P_d}}\right)+\left(\frac{\sigma_{P_d}}{a_{P_d}}a_{2i}\right)^2\Bigg(T_1-t-\frac{2}{a_{P_f}}+\frac{2\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}+$$

$$+\frac{1}{2a_{P_f}}-\frac{\exp\left\{-2a_{P_f}(T_1-t)\right\}}{2a_{P_f}}\Bigg)+2a_{2i}a_{3i}\sigma_X\frac{\sigma_{P_f}}{a_{P_f}}\left(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}\right)+\sigma_X^2a_{3i}^2\left(T_1-t\right)\Bigg],\ i=1,2,3.$$

Below are the calculations of the integrals from the Section 7:

$$J_0=-2\Bigg(\left(\sigma_{P_d}/a_{P_d}\right)^2\Bigg(T_1-t-\frac{\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_2-t)\right\}}{a_{P_d}}-\frac{1}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_1-t)\right\}}{a_{P_d}}+$$

$$+\frac{\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{2a_{P_d}}-\frac{\exp\left\{-a_{P_d}(T_2+T_1-2t)\right\}}{2a_{P_d}}\Bigg)+\left(\sigma_{P_d}/a_{P_d}\right)^2\Bigg(T_1-t-\frac{2\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{a_{P_d}}+$$

$$+\frac{2\exp\left\{-a_{P_d}(T_2-t)\right\}}{a_{P_d}}+\frac{\exp\left\{-2a_{P_d}(T_2-T_1)\right\}}{2a_{P_d}}-\frac{\exp\left\{-2a_{P_d}(T_2-t)\right\}}{2a_{P_d}}\Bigg)+$$

$$+\frac{2\sigma_{P_d}\sigma_{P_f}}{a_{P_d}a_{P_f}}\rho_{P_d,P_f}\Bigg(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}+\frac{1}{a_{P_d}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_d}}+\frac{1}{a_{P_d}+a_{P_f}}-$$

$$-\frac{\exp\left\{-\left(a_{P_d}+a_{P_f}\right)(T_1-t)\right\}}{a_{P_d}+a_{P_f}}\Bigg)-\left(\sigma_{P_f}/a_{P_f}\right)^2\Bigg(T_1-t-\frac{2}{a_{P_f}}+\frac{2\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}+\frac{1}{2a_{P_f}}-$$

$$-\frac{\exp\left\{-2a_{P_f}(T_1-t)\right\}}{2a_{P_f}}\Bigg)-2\sigma_X\rho_{P_d,X}\frac{\sigma_{P_f}}{a_{P_f}}\left(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}\right)-\sigma_X^2\left(T_1-t\right)\Bigg],$$

$$J_i=\left(\frac{\sigma_{P_d}}{a_{P_d}}a_{1i}\right)^2\left(T_1-t-\frac{2\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{a_{P_d}}+\frac{2\exp\left\{-a_{P_d}(T_2-t)\right\}}{a_{P_d}}+\frac{\exp\left\{-2a_{P_d}(T_2-T_1)\right\}}{2a_{P_d}}-\right.$$
$$\left.-\frac{\exp\left\{-2a_{P_d}(T_2-t)\right\}}{2a_{P_d}}\right)-\frac{2a_{1i}a_{2i}}{a_{P_d}a_{P_f}}\sigma_{P_d}\sigma_{P_f}\left(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}-\frac{\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{a_{P_d}}+\right.$$
$$\left.+\frac{\exp\left\{-a_{P_d}(T_2-t)\right\}}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{a_{P_d}+a_{P_f}}-\frac{\exp\left\{-a_{P_d}(T_2-t)-a_{P_f}(T_1-t)\right\}}{a_{P_d}+a_{P_f}}\right)-$$
$$-2a_{1i}a_{3i}\sigma_X\frac{\sigma_{P_d}}{a_{P_d}}\left(T_1-t-+\frac{\exp\left\{-a_{P_d}(T_2-T_1)\right\}}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_2-t)\right\}}{a_{P_d}}\right)+$$
$$+\left(a_{2i}\frac{\sigma_{P_f}}{a_{P_f}}\right)^2\left(T_1-t-\frac{2}{a_{P_f}}+\frac{2\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}+\frac{1}{2a_{P_f}}-\frac{\exp\left\{-2a_{P_f}(T_1-t)\right\}}{2a_{P_f}}\right)+$$
$$+2a_{2i}a_{3i}\sigma_X\frac{\sigma_{P_f}}{a_{P_f}}\left(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}\right)+(\sigma_X a_{3i})^2(T_1-t),\quad i=1,2,3.$$

$$Q_0=\left(\frac{\sigma_{P_d}}{a_{P_d}}\right)^2\left(T_1-t-\frac{2}{a_{P_d}}+\frac{2\exp\left\{-a_{P_d}(T_1-t)\right\}}{a_{P_d}}+\frac{1}{2a_{P_d}}-\frac{\exp\left\{-2a_{P_d}(T_1-t)\right\}}{2a_{P_d}}\right)-$$
$$-2\rho_{P_d,P_f}\frac{\sigma_{P_d}\sigma_{P_f}}{a_{P_d}a_{P_f}}\left(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}-\frac{1}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_1-t)\right\}}{a_{P_d}}+\frac{1}{a_{P_d}+a_{P_f}}-\right.$$
$$\left.-\frac{\exp\left\{-(a_{P_d}+a_{P_f})(T_1-t)\right\}}{a_{P_d}+a_{P_f}}\right)-2\sigma_X\rho_{P_d,X}\frac{\sigma_{P_d}}{a_{P_d}}\left(T_1-t-\frac{1}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_1-t)\right\}}{a_{P_d}}\right)+$$
$$+\left(\frac{\sigma_{P_f}}{a_{P_f}}\right)^2\left(T_1-t-\frac{2}{a_{P_f}}+\frac{2\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}+\frac{1}{2a_{P_f}}-\frac{\exp\left\{-2a_{P_f}(T_1-t)\right\}}{2a_{P_f}}\right)+$$
$$+2\sigma_X\rho_{P_f,X}\frac{\sigma_{P_f}}{a_{P_f}}\left(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}\right)+\sigma_X^2(T_1-t).$$

$$Q_i=\left(\frac{a_{1i}\sigma_{P_d}}{a_{P_d}}\right)^2\left(T_1-t-\frac{2}{a_{P_d}}+\frac{2\exp\left\{-a_{P_d}(T_1-t)\right\}}{a_{P_d}}+\frac{1}{2a_{P_d}}-\frac{\exp\left\{-2a_{P_d}(T_1-t)\right\}}{2a_{P_d}}\right)-\frac{2a_{1i}a_{2i}\sigma_{P_d}\sigma_{P_f}}{a_{P_d}a_{P_f}}\times$$
$$\times\left(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P_f}}-\frac{1}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_1-t)\right\}}{a_{P_d}}+\frac{1}{a_{P_d}+a_{P_f}}-\frac{\exp\left\{-(a_{P_d}+a_{P_f})(T_1-t)\right\}}{a_{P_d}+a_{P_f}}\right)-$$

$$-\frac{2a_{1i}a_{3i}\sigma_X\sigma_{P_d}}{a_{P_d}}\left(T_1-t-\frac{1}{a_{P_d}}+\frac{\exp\left\{-a_{P_d}(T_1-t)\right\}}{a_{P_d}}\right)+\left(\frac{a_{2i}\sigma_{P_f}}{a_{P_f}}\right)^2\left(T_1-t-\frac{2}{a_{P_f}}+\frac{2\exp\left\{-a_{P_f}(T_1-t)\right\}}{a_{P-f}}+\right.$$

$$\left.+\frac{1}{2a_{P_f}}-\frac{\exp\left\{-2a_{P_f}(T_1-t)\right\}}{2a_{P_f}}\right)+\frac{2a_{2i}a_{3i}\sigma_X\sigma_{P_f}}{a_{P_f}}\left(T_1-t-\frac{1}{a_{P_f}}+\frac{\exp\left\{\ \right\}^{-a_{P_f}(T_1-t)}}{a_{P_f}}\right)+(\sigma_X a_{3i})^2(T_1-t),\quad i=1,2,3.$$

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Расчет выпуклой поправки к форвардным ставкам в модели Васичека для экзотических форвардных контрактов

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Аннотация. В данной статье мы рассмотрели оценку форвардных контрактов, которые являются популярными финансовыми инструментами для покупки или продажи каких-либо активов в заданный момент времени в будущем по указанной фиксированной цене. Условия таких контрактов могут устанавливаться в зависимости от потребностей покупателей или продавцов, а торговля ими происходит на внебиржевом рынке. Это отличает их от фьючерсов, которые торгуются на бирже на стандартизированных условиях. Фокусом нашего исследования являются форвардные контракты на процентную ставку с выплатой в момент фиксации плавающей ставки (in-arrears forward rate agreement, или in-arrears FRA). Они отличаются от обычных форвардных контрактов на ставку тем, что плавающая процентная ставка выплачивается в момент фиксации. Мы рассчитали выпуклую поправку к плавающей процентной ставке, возникающую в таких контрактах, при различных конфигурациях времени выплат в однофакторной стохастической модели Васичека. С помощью принципа безарбитражности мы показали, что поправка будет неотрицательной в случае, когда выплаты происходят до конца периода начисления, и отрицательной в случае, когда выплаты происходят после. Мы также изучили in-arrears форвардные и опционные контракты, в которых ставка и номинал, на который начисляется эта ставка, относятся к разным валютам quanto in-arrears FRA и quanto in-arrears опционы). Мы убедились, что quanto in-arrears FRA равен обычному in-arrears FRA в случае, когда валюты совпадают, и что quanto in-arrears опционы дороже обычных.

Ключевые слова: выпуклая поправка; форвардный контракт на процентную ставку (FRA); модель Васичека; принцип безарбитражности; форвардный контракт на процентную ставку с мгновенной выплатой (in-arrears FRA); кванто-форвардный контракт на процентную ставку с мгновенной выплатой (quanto in-arrears FRA); LIBOR; MOSPRIME, форвардный контракт на процентную ставку с мгновенной выплатой (in-arrears FRA / iFRA).

Классификация JEL: G12, G13, C02.

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