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ТЕОРЕТИЧЕСКИЕ И МЕТОДОЛОГИЧЕСКИЕ ПРОБЛЕМЫ

Evolutionary nonstationarity of economic cycles

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Abstract. In the article, the nonstationarity of economic cycles is studied using their one-dimensional model of the "investment → income" type. The model interprets the cycle as random oscillations of an elastic system induced by exogenous (investment fluctuations) and endogenous (system properties) causes. This approach provided a quantitative description of economic cycles through the parameters of the elastic system — its natural frequency and damping factor. The nonstationarity of cycles is analyzed by the time trend of their natural frequencies. Such an analysis was performed for the period 1960–2020 by the amplitude spectra of US GDP deviations. Its results showed a simultaneous and steady decrease in the duration of the three considered cycles. This means that the results of observing these cycles do not have the ergodic property. Therefore, the adaptation of the cycle model to empirical data is possible for a time interval in which it can be considered pseudo-stationary.

JEL Classification: C01, C02, C13, C26.

Keywords: economic cycle, random oscillations, elastic system, Fourier transform.

UDC: 330.43.

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1. INTRODUCTION

In the theory of stochastic dynamics of economic cycles (Karmalita, 2020), the model of the cycle $\Xi(t)$ with period T_0 is a second-order ordinary differential equation:

$$\ddot{\Xi}(t)+2h\dot{\Xi}(t)+(2\pi f_0)^2\Xi(t)=E(t). \qquad (1)$$

The above model describes random oscillations induced by a linear elastic system with natural frequency $f_0 = 1/T_0$ and damping factor h under the white noise $E(t)$ (Bolotin, 1984). Equation (1) made it possible to move from modeling the results of observations of economic cycles to modeling their mechanism. According to model (1), income oscillations $\Xi(t)$ are due to both investment fluctuations $E(t)$ and properties of an elastic system. Note that equation (1) not only explains the cycle mechanism, but also provides its quantitative description in terms of the parameters h and f_0.

Technological and managerial progress in the economy must lead to a change in the characteristics of economic systems, that is, a change in cycle parameters. We start by looking at the damping factor h which characterizes the efficiency of the elastic system. In the theory under consideration, the root-mean-square (*rms*) gain was proposed as an indicator of the economic system's efficiency in the form of the following ratio: $K_\sigma = \sigma_\xi / \sigma_\varepsilon$. Here σ_ξ and σ_ε are *rms* values of income (output) and investment (input) functions, respectively. Since both K_σ and h are related to the efficiency of the system under consideration, there is a quantitative relationship between them. With a discrete representation of the income by a dimensionless time interval ($\Delta t = 1$), the latter has the form shown in Fig. 1 (Karmalita, 2020).

Consequently, the evolutionary increase in the efficiency of economic systems is *a priory* accompanied by a decrease in the value of h.

Within the framework of the above theory, the natural frequency f_0 of the cycle correlates with the inclusive wealth W_I of corresponding economic system:

$$f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{w_s}{W_I}}. \qquad (2)$$

Here W_I is the monetary value of all system assets (production, labour, inventory, and finance), and w_s is the dynamic factor that characterizes the system's ability to withstand investment fluctuations and eliminate their consequences. The evolutionary development is accompanied by an increase in the inclusive wealth of systems, which is illustrated in Fig. 2 (Yamaguchi, Islam, Managi, 2019).

A viewed increase in the total wealth of economic systems, as follows from (2), should lead to a decrease in natural frequency f_0. Since the dimension of the coefficient w_s is equal to \$/year^2, it can be interpreted as the acceleration with which the inclusive wealth is redistributed among its assets when the structure of W_I changes under the influence of investment fluctuations. Conceptually, w_s must grow due to evolutionary progress in the development of methods and tools for transforming one type of wealth asset into another. Thus, both terms of expression (2) grow, and the parameter f_0 will change over time. But the nature of this change (growth—decrease) will be determined by the dominance of the change in the numerator or denominator of expression (2). The purpose of this article is a quantitative analysis of the actual change in the natural frequency of known economic cycles. To do this, it is necessary to start with the choice of informative indicators, and then evaluate their trends over time using the available econometric data.

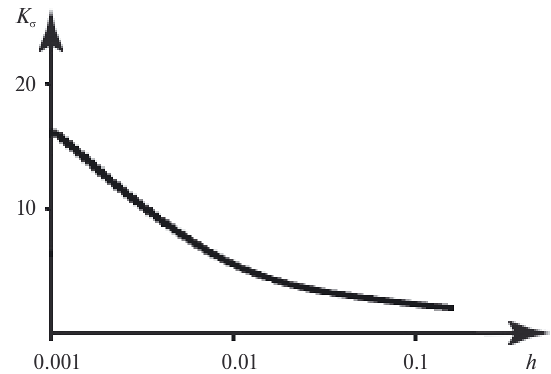

Fig. 1. Dependence of K_σ on h

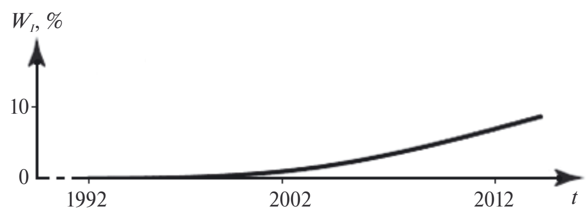

Fig. 2. Evolutionary growth of W_I (per capita)

2. FORMALIZATION OF THE PROBLEM

The random nature of investments inherent in a market economy made it possible in (Karmalita, 2020) to formally represent them as the sum of a deterministic trend and stochastic fluctuations $E(t)$. Such a presentation correlates to Cooley and Prescott's postulate (Cooley, Prescott, 1995) of the need to consider economic growth and fluctuations together. Accordingly, the income function $X(t)$ can also be written as the sum of two terms: $X(t) = L(t) + \delta X(t)$, where $L(t)$ is the long-term trend in income due to the investment trend $M(t)$, and the deviations $\delta X(t)$ are caused by investment fluctuations $E(t)$. Deviations $\delta X(t)$ include all known economic cycles (Kondratiev, Kuznets, Juglar, and Kitchin) as well as other possible income fluctuations, that is:

$$\delta X(t) = \Xi_{Ko}(t) + \Xi_{Ku}(t) + \Xi_J(t) + \Xi_{Ki}(t) + \ldots$$

In the time domain, the relationship between investment fluctuations $E(t)$ and a given income cycle $\Xi(t)$ with natural frequency f_0 is described by equation (1). The properties of the linear elastic system in the frequency domain are described by its amplitude-frequency characteristics $A(f)$ (Fig. 3).

$A(f)$ determines the ratio of the amplitudes of the input (investment) and output (income) harmonics. In accordance with the shape of the characteristic $A(f)$, the values of random oscillations $\Xi(t)$ are determined mainly by harmonics in the frequency band $f_1 = 0.7f_0 \le f \le f_2 = 1.4 f_0$. In other words, economic cycles are narrow-band random processes.

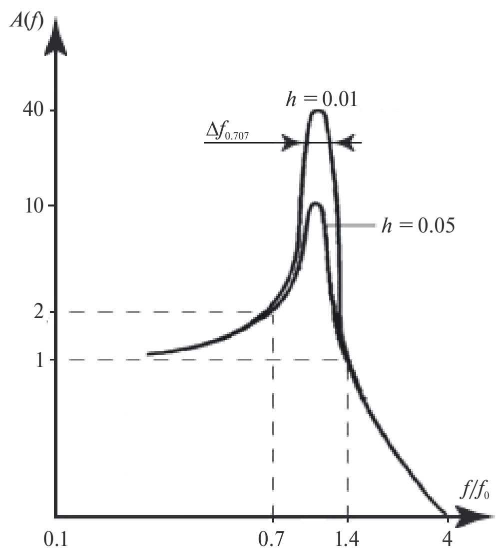

Fig. 3. Amplitude frequency characteristics of the linear elastic system

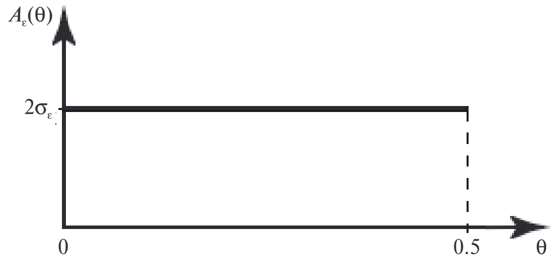

Fig. 4. Amplitude spectrum of white noise

The peak of $A(f)$ corresponds to the frequency $f_h = \sqrt{f_0^2 - (h/2\pi)^2}$, which is called the damping-corrected natural frequency of the system. The parameter h determines the width of band $\Delta f_{0.707}$ of the characteristic $A(f)$ at points where $A(f) = 0.707A(f_h)$, as shown in Fig. 3. The relationship between $\Delta f_{0.707}$ and the damping factor h is as follows (Pain, 2005):

$$h = \pi\Delta f_{0.707}. \tag{3}$$

Since the amplitude spectrum of the output of a linear dynamic system is equal to the product of its amplitude-frequency characteristics and the amplitude spectrum of the input, then $A_\Xi(f) = A(f)A_E(f)$. Note, that white noise has a uniform spectrum (Fig. 4). Here $\sqrt{D_E} = \sigma_E$ is the root-mean-square value of investment fluctuations, and θ is the relative frequency corresponding to the dimensionless time sampling interval $\Delta t = 1$. Therefore, $A_\Xi(f)$ is directly proportional to the amplitude frequency characteristics of the system due to the uniformity of $A_E(f)$. Therefore, estimates of $\theta_h = \sqrt{\theta_0^2 - (h\ \ t/2\pi)^2}$ can be determined from the amplitude spectrum of oscillations $\Xi(t)$.

3. ESTIMATING THE CYCLE PARAMETERS

In econometric studies, to quantify the income function $X(t)$ the gross domestic product (GDP), hereinafter $G(t)$, is usually used. First, we need to assess the suitability of GDP estimates for evaluating the values of the parameters f_0 and h. Recall that the value of GDP is a monetary estimate of manufactured goods and provisioned services for a certain period ΔT. The income function can be represented as the sum of its trend $L(t)$ and deviations $\delta X(t)$. Therefore, $G(t)$ is mathematically described in the following form:

$$G(t) = \int_{t-\Delta T}^{t} X(t)dt = \int_0^t k(\tau)X(t-\tau)d\tau\ ,$$

$$\int_0^t k(\tau)L(t-\tau)d\tau + \int_0^t k(\tau)\delta X(t-\tau)d\tau = G_L(t)\ + g(t). \tag{4}$$

In other words, the GDP function can be interpreted as the result of measurements of the income function using an estimator the inertial properties of which are described by the impulse response (*IR*) function $k(\tau)$:

$$k(\tau) = \begin{cases} 1, & 0 \le \tau \le \Delta T; \\ 0, & \tau < 0; \tau > \Delta T. \end{cases}$$

Applying the linear operator (4) to a homogeneous version of equation (1) leads it to the following form: $\ddot{g}_\Xi(t) + 2h\dot{g}_\Xi(t) + (2\pi f_0)^2 g_\Xi(t) = 0$. Hence it follows that the parameters f_0 and h also characterize the properties of the corresponding GDP estimates.

In the frequency domain, equation (4) transforms into the multiplication of the corresponding Fourier images: $G(f) = G_L(f) + g(f) = H(f)L(f) + H(f)\delta X(f)$. The operator $H(f)$, being the Fourier transform of $k(\tau)$, has the following form (Pavleino, Romadanov, 2007):

$$H(f) = \int_{-\infty}^{\infty} k(t)e^{-j2\pi ft}\,dt =$$

$$= \Delta T\ \mathrm{sinc}(\pi\Delta Tf)e^{-j\pi\Delta Tf} = A_k(f)\ ^{j\Phi_k(f)}.$$

Here $A_k(f)$ and $\Phi_k(f)$ are the amplitude- and phase-frequency characteristics of the estimator, and

$$\mathrm{sinc}(\pi\Delta Tf) = \begin{cases} 1, & f = 0; \\ (\sin\pi\Delta Tf)/(\pi\Delta Tf), & f \ne 0. \end{cases}$$

The amplitude-frequency characteristics $A_k(f)$ is shown in Fig. 5.

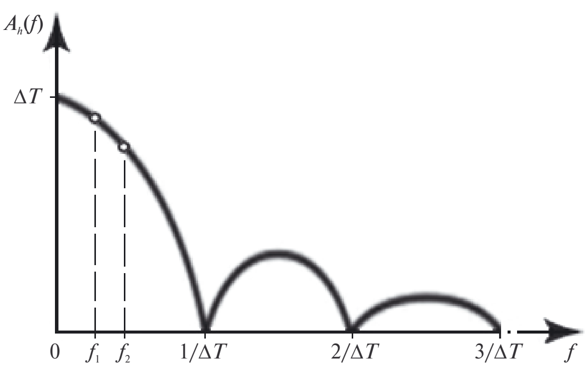

Fig. 5. The frequency characteristic of the estimator

From its form it follows that the estimator has a different effect on the harmonics of the cycle with frequency $f_0 = 1/T_0$, which are concentrated in frequency band $f_1, ..., f_2$. Moreover, this effect intensifies with

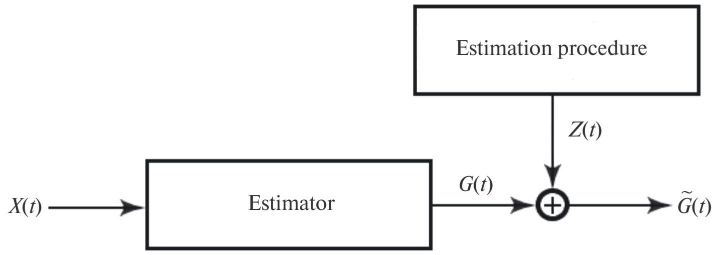

Fig. 6. Diagram of estimating GDP

increasing f_0 due to growing the slope of $A_k(f)$. Let us estimate this effect using the example of the Kitchin cycle, which has the highest frequency. In the case of using quarterly GDP estimates, the value $\Delta t = \Delta T = 0.25$ years.

For such an illustration, we will take the estimate $T_{Ki} \approx 3.6$ years (Karmalita, 2023). Consequently, the natural frequency of the Kitchin cycle is $f_{Ki} = 1/T_{Ki} \approx 0.28$ cycle/year, the values $f_1 = 0.7 f_0 \approx 0.19$ and $f_2 = 1.4 f_0 \approx 0.39$. In this case, the AF characteristics of the estimator has values $A_k(f_1) = \sin(0.2 \times 0.25\pi)/(0.2 \times 0.25\pi) \approx 0.996$, $A_k(f_{Ki}) \approx 0.991$, and $A_k(f_2) \approx 0.984$. The difference between the values of $A_k(f_1)$ and $A_k(f_2)$ from $A_k(f_{Ki})$ does not exceed $\pm 0.7\%$. Given the steepness of characteristic $A(f)$ (see Fig. 3), we can assert that the peaks of the real spectrum of Kitchin cycle and the spectrum of GDP will coincide. Therefore, the values f_h for cycles of longer duration will certainly be defined as the frequency coordinates of their peaks in the image $g(f)$.

Let us turn to the procedure for estimating GDP, conditionally presented in Fig. 6. The diagram presents the error $Z(t)$ describing the actual properties of the estimation procedure. This error arises, for instance, due to unreliable statistical data, its incompleteness, and the performer's skill. Therefore, the available data related to the values of GDP are the estimates $\tilde{G}(t)$ determined as: $\tilde{G}(t) = G(t) + Z(t) = \tilde{G}_L(t) + \tilde{g}(t)$. Here and further in the text, the sign "$\sim$" denotes a quantitative estimate of the noted variable.

The function $Z(t)$ (measurement noise) is usually modeled as "white" noise, which does not affect the position of the peak of the spectrum $A_\Xi(f)$ on the frequency axis. However, the sharpness of spectrum $A(f)$, presented in Fig. 3, makes the white noise assumption not strict. Conceptually, the transition to correlated measurement noise leads to the appearance of local gradients in the uniform spectrum of white noise. In order for noise $Z(t)$ to influence the position of the peak of curve $A_\Xi(f)$, the gradient of the noise spectrum must dominate in the vicinity (less than $\Delta f_{0.707}$) of the value θ_h. Moreover, the shift of the spectrum peak along the frequency axis leads to a violation of the symmetry of spectrum $A_\Xi(f)$, which is detected both numerically and visually.

Very often in econometric quarterly estimates of GDP are used. It should be noted that they are characterized by a non-equidistance sampling interval due to the different duration of quarters during the year. For the first quarter, Δt_1 is 90 or 91 (leap year) days, $\Delta t_2 = 91$, $\Delta t_3 = 92$, and $\Delta t_4 = 92$. Such a change in the sampling interval leads to additional randomization of income oscillations $\Xi(t)$. As a result, the slope of the peak of the amplitude spectrum becomes flatter, which leads to an increase in the estimate of h. Obviously the cycle with the highest natural frequency will have the largest bias in the damping factor estimate. Numeric simulations show that above randomizing a harmonic ($h = 0$) with the Kitchin cycle frequency leads to the estimate $\tilde{h} < 0.002$. This bias can be considered insignificant for econometric studies. As for the position of the cycle peak on the frequency axis, it does not change.

Thus, the choice of estimates of the parameter θ_h as a quantitative measure of the nonstationarity of cycles seems justified. In this case, the research method will be Fourier analysis, and the empirical data will be quarterly estimates of USA GDP.

4. RESEARCH RESULTS

Estimates of the damping-corrected natural frequencies of the cycles were determined using economic data for the period 1960–2020 (Fig. 7).

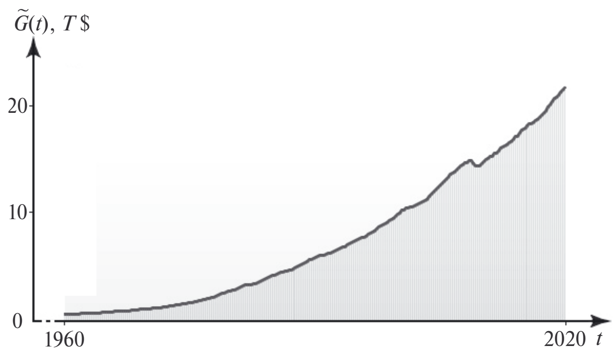

Fig. 7. Real GDP estimates of the US economy

Source: Federal Reserve Economic Data (2023) (https://fred.stlouisfed.org).

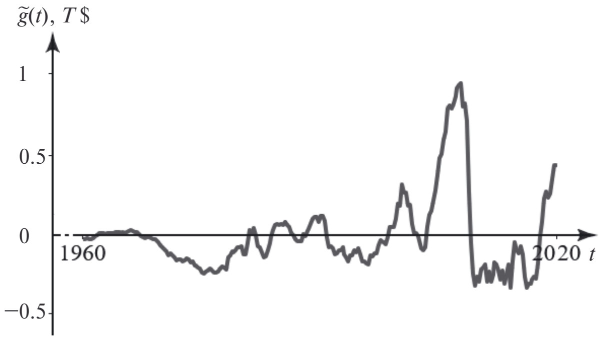

Fig. 8. Deviations of GDP estimates

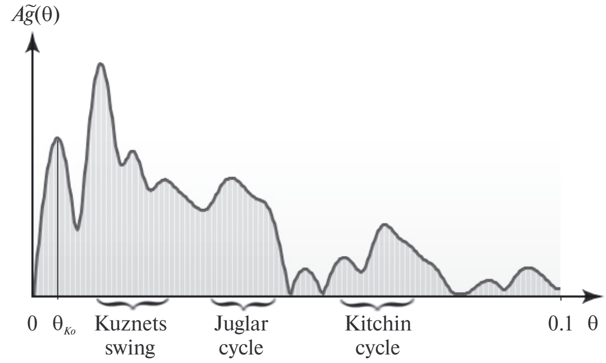

Fig. 9. The spectrum of GDP deviations

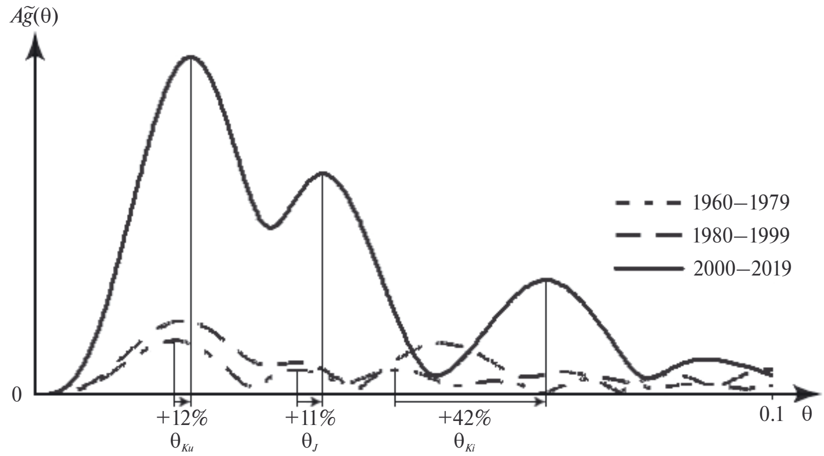

Fig. 10. Trends of natural frequencies of economic cycles

The sampling interval was $\Delta t \approx 0.25$ years, so the number of samples $n = 240$. Fourier analysis must be applied to GDP deviations $\tilde{g}_i = \tilde{g}(t_i) = \tilde{g}(\Delta t\, i)$, which should be separated from $\tilde{G}_i$ using the trend $\tilde{G}_{Li}$. Since the latter is unknown, its estimates $\hat{G}_{Li}$ were determined by the least squares method (Brandt, 2014). After that, estimates of GDP deviations were calculated (Fig. 8) as $\tilde{g}_i = \tilde{G}_i - \hat{G}_{Li}$ relative to the regression $\tilde{G}_{Li} = 0.365i^2 - 0.535i + 564\ (i = 0,\ 239)$.

The amplitude spectrum of the deviations was determined through the Fourier transform (Cho, 2018). Since we used the dimensionless sampling interval $\Delta t = 1$, the results will be presented in terms of relative frequency $0 \le \theta \le 0.5$. The amplitude frequency spectrum $A_{\tilde{g}}(\theta)$ of GDP deviations is shown in Fig. 9.

There is only one period of Kondratiev wave in an interval of 60 years, so for this cycle it was possible to unambiguously determine the relative frequency $\tilde{\theta}_{Ko} \approx 0.0043$ and period $\tilde{T}_{Ko} = \Delta t / \tilde{\theta}_{Ko} = 0.25 / 0.0043 \approx 58$ years. The frequency values were determined with a resolution $\Delta\theta \approx 0.00024$. An estimate of the Kondratiev damping factor can be found as $\tilde{h}_{Ko} \approx \pi \Delta\theta\,(0.707 / 0.25) \approx 0.058$.

The observed stratification of the Kuznets swing and the Kitchin cycle, as well as the deformation of the Juglar spectrum, is indirect indicators' package of the change in their natural frequencies over time. To determine a stable trend in a time series, we need to have at least three values, so the estimates of θ_0 were calculated on a time base of 20 years. This approach was applied to fragments of the initial data $\tilde{G}_i$, using for the period 1960–1979 regression $\tilde{G}Li = 0.367i^2 - 3.85i + 610$, for $1980-1999-\tilde{G}Li = 0.359i^2 + 57.8i + 2843$, and for $2000-2019-\tilde{G}Li = 0.391i^2 + 108.8i + 10\,166$. The results of the research are shown in Fig. 10.

The figure above shows a simultaneous steady increase (decrease) in the frequencies (durations) of the three considered cycles. Estimates of the relative frequencies (durations in years) of cycles increased (decreased) for:

- Kuznets swing — from 0.018 to 0.02 (13.7 → 12.3),
- Juglar cycle — from 0.034 to 0.038 (7.3 → 6.6),
- Kitchin cycle — from 0.048 to 0.067 (5.3 → 3.7).

The increase of the natural frequencies of the cycles is explained by the dominant growth of the dynamic factor w_s. Recall that w_s were interpreted as the acceleration at which inclusive wealth is redistributed among its assets. Its growth is due to significant progress in the development of methods and tools for converting one type of asset of the economic system into another.

5. CONCLUSIONS

The time trends of natural frequencies of economic cycles presented in empirical data of a market economy are studied. Fourier analysis of deviations of US GDP showed a simultaneous steady decrease in the duration of the three considered cycles over time. Thus, the hypothesis expressed in (Karmalita, 2020) about the evolutionary change in the duration and intensity of economic cycles is confirmed. Therefore, the economic systems are, in principle, nonstationary, and the results of observing their behavior do not have the property of ergodicity.

As a rule, a probabilistic description of nonstationary random processes is possible only on a set of their realizations. In fact, economic data, being essentially a chronicle, is available in a single implementation. So, estimates of cycle parameters can be evaluated on pseudo-stationary fragments of the cycle trajectory, in which evolutionary changes in parameters do not exceed the statistical uncertainty of their estimates. An example of selecting such fragments by testing the hypothesis about the homogeneity of cycle correlations using statistical inference methods exists (Karmalita, 2020).

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Эволюционная нестационарность экономических циклов

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Аннотация. Статья посвящена исследованию нестационарности экономических циклов, описываемых одномерной моделью, вход которой — «инвестиции», а выход — «доходы». Цикл рассматривается как случайные колебания упругой системы, вызванные внешними (колебания инвестиций) и внутренними (свойства системы) факторами. Такой подход позволил дать количественное описание экономических циклов через параметры упругой системы: собственную частоту и коэффициент затухания. Нестационарность циклов оценивалась по поведению собственных частот во времени. В качестве эмпирических данных был выбран ВВП США за период 1960–2020 гг. Амплитудные спектры циклов вычислялись методом дискретного преобразования Фурье разности между значениями ВВП и его квадратичного тренда, взятых с шагом в один квартал. Результаты спектрального анализа показали одновременное и устойчивое снижение продолжительности трех рассматриваемых циклов, на основании чего был сделан вывод о неэргодичности экономических циклов. Поэтому адаптация модели цикла к эмпирическим данным возможна лишь на временных интервалах, где ее можно считать псевдостационарной.

Классификация JEL: C01, C13, C02, C26.

Ключевые слова: экономический цикл, случайные колебания, упругая система, преобразование Фурье.

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