
ТЕОРЕТИЧЕСКИЕ
И МЕТОДОЛОГИЧЕСКИЕ ПРОБЛЕМЫ

Production theory for constrained linear activity models¹

© 2023 S. Lahiri

S. Lahiri,

Lok Jagruti (LJ) University, India; e-mail: somdeb.lahiri@gmail.com

Received 24.09.2022

Abstract. The purpose of this paper is to generalize the framework of activity analysis discussed in the paper by Antonio Villar without claiming any dimensional requirements on the activity matrices and by introducing a model of activity analysis in which each activity may (or may not) have a capacity constraint. We follow the usual nomenclature of input-output analysis for “the quantity of a good supplied to the consumers outside the production (or manufacturing) sector” and refer it as “final demand”. We obtain results similar to those in Villar concerning solvability, non-substitution and existence of efficiency prices. We apply our analysis and results to the two-period multisector activity analysis model with capacity constraints. The activity matrix is the difference between a non-negative output coefficient matrix and a non-negative input coefficient matrix, with the coefficients being measured in money units for each activity. Almost all the results obtained thus far get replicated in this macroeconomic context. However, some reformulations are required for issues related to existence of equilibrium price vector and as a consequence, issues related to efficiency prices via the non-substitution theorems. The corresponding concepts in this application refer to “inflation rate” vectors.

Keywords: constrained, linear activity analysis, solvability, non-substitution theorem, efficiency prices.

JEL Classification: C67, E16, E23, E31.

For reference: **Lahiri S.** (2023). Production theory for constrained linear activity models. *Economics and Mathematical Methods*, 59, 1, 5–15. DOI: 10.31857/S042473880024866-1

This paper is in honour of Professor Dipankar Dasgupta and Professor Pradip Maiti, who taught me Linear Production Models and Linear Programming respectively.

1. INTRODUCTION

The purpose of this paper is to generalize the framework of activity analysis discussed in A. Villar (Villar, 2003) and obtain similar results concerning solvability. We generalize the model due to A. Villar (Villar, 2003), without requiring any dimensional requirements on the activity matrices and by introducing a model of activity analysis in which each activity may (or may not) have a capacity constraint i.e., a maximum level at which the activity can operate. This may be one way to accommodate meaningful non-linearities similar to that considered for input–output analysis in I. Sandberg (Sandberg, 1973).

In I. Sandberg (Sandberg, 1973), the input-output coefficients were assumed to be differentiable, which is very likely an approximation of the more realistic and practically applicable representation in which the input-output coefficients are piecewise constant. Minor, technical increments on the solvability result in Sandberg (Sandberg, 1973) can be found in P. Chander (Chander, 1983) and some references therein. Undoubtedly, piecewise constant input-output constants are more difficult to deal with mathematically than differentiable ones. Non-linearities are much easier to represent in the framework of linear activity analysis by introducing upper bounds — whenever there is a capacity constraint for the levels at which each such activity may operate. We do this by introducing a model of activity analysis in which each activity may (or may not) have a capacity constraint. In this paper we follow the usual nomenclature of input-output analysis for “the quantity of a good supplied to the consumers outside the production (or manufacturing) sector” and refer it as “final demand”.

It seems that in this significantly more general framework we are able to obtain the desired results concerning solvability and existence of an equilibrium price-vector under weaker assumptions than the corresponding requirements in A. Villar (Villar, 2003). The property that guarantees solvability has the following

¹ This paper generalizes and extends the framework of analysis as well as the results in an earlier paper by the author entitled “Production Analysis for Proper Activity Matrices” (<https://drive.google.com/file/d/15E2SI1b5NeWBgtRI-O1gb1UA9CG6QhJc/view>).

economic interpretation: Given two price vectors p, q a final demand vector and unit prices of operating capacity constrained activities, if the revenue from operating every unconstrained activity at unit level at price-vector p is no less than doing the same at price-vector q and if the revenue from operating a constrained activity at unit level at price-vector p is no less than doing the same at price-vector q plus the unit price of operating the activity then the value of the final demand at price vector p is no less than the value of the final demand at price-vector q plus the total cost of the capacities. In the context of the constraint linear activity analysis model, we call this property “weakly proper”. We also prove a version of the Non-Substitution Theorem that establishes the existence of “efficiency price-vectors” as a joint product. However, our Non-Substitutions Theorem — in spite of the generality of our model as compared to the one due to A. Villar (Villar, 2003) — requires that if there are capacity constraints, then there is a minimal subset of the set of capacity constrained activities that are always used up to full capacity, for the production of all producible final demand vectors. Further, these capacity constrained activities are the only ones whose capacities are binding for some producible final demand vector. Hence there is a clear dependence of the equilibrium price vector on the final demand vector (i.e. the vector of quantities supplied for non-manufacturing prices); unlike the conclusion of Sraffian economics², and we are able to show this without using production functions. Our framework is a generalization of the one used in Sraffian economics.

In a concluding section of this paper, we apply our analysis and results to the two-period multisector activity analysis model with capacity constraints. Production takes place in the current period and the sale of goods outside the production system, takes place in a subsequent period. The activity matrix is the difference between a non-negative output coefficient matrix and a non-negative input coefficient matrix, with the coefficients being measured in money units for each activity. Almost all the results obtained thus far get replicated in this macroeconomic context. However, some reformulations are required for issues related to existence of equilibrium price vector and as a consequence, issues related to efficiency prices via the non-substitution theorems. The corresponding concepts in this application refer to “inflation rate” vectors, i.e., “equilibrium inflation rate-vector” and “efficiency equilibrium rate-vector” respectively. In the case of final demand for services (provided by the service sector) which do not require manufacturing, but are nonetheless measured in current producer prices, this may imply a difference between producer prices and the prices that consumers are required to pay for the services during the current period.

In what follows we will be making extensive use of mathematical results in Topics 2 and 3 — and therefore by implication results in Topic 1 — of Lahiri (Lahiri, 2022). Sometimes, when there is no scope for confusion, given two vector/points x and y in a real Euclidean space of the same dimension we use $x \geq y$ to denote that every co-ordinate of x is **greater than or equal to** the corresponding coordinate of y , $x > y$ to denote $x \geq y$, but $x \neq y$ (x is not equal to y), and $x \gg y$ to denote every coordinate of x is **strictly greater than** the corresponding coordinate of y .

2. MOTIVATION

Consider a very simple production process which produces a single output (“corn”) from a single input (once again “corn”). In the classical or Leontief Input-Output (IO) Model one assumes that there exists a **fixed positive constant** a such that ax units of corn are **required to produce** x units of corn. The production process is productive **if and only if** $0 < a < 1$. However, in reality the assumption of a fixed input-out coefficient a is unrealistic. Fertility of the soil is not uniform. Hence, it is unreasonable to assume that input-output coefficient remains constant for all levels of output.

Sandberg (Sandberg, 1973) suggested that the input-output coefficient(s) is(are) a differentiable function(s) of the gross output. Once again, differentiability of input-output coefficient appears to be an “unrealistic” assumption for actual production processes. Even if such an assumption is theoretically true, it is extremely difficult to invoke it for practical use in “production planning”. It is perhaps more realistic to assume that input-output coefficients are piece-wise constant. In what follows we generalize such an assumption and develop the theory that follows from it to the case of production with possibly more than one good.

The Sraffian conclusion of market prices being independent of the quantity “supplied” (which is referred to in the literature on input-output analysis as “final demand”) with fixed input-output coefficients and hence fixed unit costs of production with competitive factor markets, is not unreasonable at all in the “one good” setting, since that is precisely what is indicated by the intersection of any demand curve with a horizontal marginal cost (or perfectly competitive supply curve). However, in the more general setting with piece-wise

² See the sixth paragraph at: <https://www.rethinkeconomics.org/journal/sraffian-economics/>

constant input-output coefficients or piece-wise constant marginal cost functions, neither would the associated extension of the Sraffian linear model indicate such independence between market price and quantity supplied and nor would it be implied by the intersection of the market demand and marginal cost curves.

3. THE MODEL

Consider an economy with m produced goods indexed by $i = 1, \dots, m$ and n activities indexed by $j = 1, \dots, n$.

An $m \times n$ matrix of real numbers is said to be non-zero, if it has “at least one non-zero entry”.

Note. The rank of a non-zero matrix is a positive integer.

Let M be a non-zero $m \times n$ matrix of real numbers called an **activity matrix** the column j of which denoted M^j for $j \in \{1, \dots, n\}$ denotes the amount of **net output** of each good if the activity j is operated at unit level. Thus for $I \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, the entry i of M^j denoted m_{ij} denotes the **net output** (of the produced good i if the activity j (or activity j) is operated at unit level. If m_{ij} is negative, then $-m_{ij}$ is the amount of the produced good i used as **net input**, if the activity j is operated at unit level.

In what follows, unless otherwise stated, we shall use net output and output interchangeably. The same applies for net input and input.

An **activity-vector** is a column vector $x \in \mathbb{R}_+^n$ such that for all $j \in \{1, \dots, n\}$, the row j (or coordinate) of x , denoted x_j , is the level at which activity j is operated.

A **constrained linear activity analysis model (CLAAM)** is a pair $(M, \langle \bar{x}_j \mid j \in W \rangle)$ where M is an $m \times n$ activity matrix and if $W \neq \emptyset$ then $\langle \bar{x}_j \mid j \in W \rangle$ is an array with $\bar{x}_j \in \mathbb{R}_{++}$ being the maximum level at which activity $j \in W \subset \{1, \dots, n\}$ can operate, the possible levels of operation for activities in $\{1, \dots, n\} \setminus W$ being unbounded above. If $W = \emptyset$, then a constrained linear activity analysis model with activity matrix M is (M, \emptyset) .

In the case of (M, \emptyset) no activity has a capacity constraint.

Note. This is a very general formulation. In particular there could be non-negative $m \times n$ matrices B, A such that $M = B - A$.

A column vector $d \in \mathbb{R}_+^m \setminus \{0\}$ is said to be a **final demand vector** if for all $i \in \{1, \dots, m\}$, row i of d , i.e. d_i represents the quantity of good supplied i for non-manufacturing/non-production purposes, i.e. quantity of the good supplied i to the consumers outside the production process.

The following is an example of an activity matrix, different from any in A. Villar (Villar, 2003) and any known to us otherwise (see chapters 6 and 7 of K. Lancaster (Lancaster, 1968)).

Example 1. Let C be a $n \times n$ matrix of non-negative real numbers such that for all $i, j \in \{1, \dots, n\}$, c_{ij} which is the element (i, j) of C , is a non-negative real number that denotes the level at which activity i is required to be operated — to produce enough of the m produced goods — so as to be able to operate activity j at unit level. Let B be a $m \times n$ matrix of non-negative real numbers such that for all $h \in \{1, \dots, m\}$ and $I \in \{1, \dots, n\}$, b_{hi} is the element (h, i) of B , denoting the (net) amount of good h that is produced with the purpose of satisfying final demand, if activity i operates at unit level.

Let $M = B(I - C)$.

If $x \in \mathbb{R}_+^n$ satisfying $x_j \leq \bar{x}_j$ for all $j \in W$, if $W \neq \emptyset$ and unconstrained otherwise, is an activity-vector then Cx denotes the level at which the activities are required to operate in order to “operationalize” the activity vector x . The remaining levels of activity vector $(I - C)x$ are used to produce the “produced goods” in the amount $B(I - C)x$, **provided** $(I - C)x \in \mathbb{R}_+^n$.

In what follows, unless otherwise stated or required, we will write $(M, \langle \bar{x}_j \mid j \in W \rangle)$ to represent a CLAAM, with the implicit understanding that if $W = \emptyset$, then the CLAAM reduces to or represents (M, \emptyset) .

The following definition will prove to be important in the analysis that follows.

A **price-vector** is a vector $p \in \mathbb{R}_+^m \setminus \{0\}$ where the coordinate i of p , denoted p_i is the unit price at which the produced good i is sold in the market. Let $w > 0$ denote wage rate of labour.

4. SOLVABILITY OF CLAAM

Given a CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$ and a non-empty subset of activities J , we define the set **Constrained Span** $(M, \langle \bar{x}_j | j \in W \rangle, J)$, denoted $CS(M, \langle \bar{x}_j | j \in W \rangle, J) = \{Mx \in \mathbb{R}^m | x \in \mathbb{R}^n \text{ with } x_j \leq \bar{x}_j \text{ for } j \in W \text{ and } x_j = 0 \text{ for all } j \notin J\}$.

Thus, $CS(M, \langle \bar{x}_j | j \in W \rangle, J) \subset \text{Span}(M, J)$.

If $J = \{1, \dots, n\}$, then $CS(M, \langle \bar{x}_j | j \in W \rangle, J)$ is written simply as $CS(M, \langle \bar{x}_j | j \in W \rangle)$.

Let J be a non-empty subset of $\{1, \dots, n\}$.

The content of the following property is based on one due to Villar (Villar, 2003).

A CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$ is said to be **weakly proper for** (a non-empty subset of activities) J , if for all $p, q \in \mathbb{R}_+^m$, any array of non-negative real numbers $\langle \alpha_j | j \in W \cap J \rangle$ and $d \in CS(M, \langle \bar{x}_j | j \in W \rangle) \cap (\mathbb{R}_+^m \setminus \{0\})$: $[p^T M^j - \alpha_j \geq q^T M^j \text{ for all } j \in W \cap J \text{ and } p^T M^j \geq q^T M^j \text{ for all } j \in J \setminus W \text{ implies } p^T d - \sum_{j \in W \cap J} \alpha_j \bar{x}_j \geq q^T d]$.

We use $(M, \langle \bar{x}_j | j \in W \rangle)$ as weakly proper for J and $(M, \langle \bar{x}_j | j \in W \rangle)$ is a CLAAM weakly proper for J , or just $(M, \langle \bar{x}_j | j \in W \rangle, J)$ is weakly proper for J interchangeably.

Weakly proper in the context of CLAAM has the following economic interpretation (if we interpret α_j as the unit cost of operating a capacity constrained activity $j \in W \cap J$):

if the revenue from operating every unconstrained activity in J at unit level at price-vector p is no less than doing the same at price-vector q and if the revenue from operating a capacity constrained activity in J at unit level at price-vector p is no less than doing the same at price-vector q **plus** the unit price of the capacity then the value of the final demand d at price vector p is no less than the value of the final demand d at price vector q **plus** the total cost of the capacities.

In the interpretation provided above we are assuming that capacities have an imputed price/shadow price given by the alphas up to the maximum that is possible.

A CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$ is said to be **weakly proper** if it is weakly proper for $\{1, \dots, n\}$.

In particular, (by setting $\alpha_j = 0$ for all $j \in W$) for all $p, q \in \mathbb{R}_+^m$ and $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap (\mathbb{R}_+^m \setminus \{0\})$: $[p^T M \geq q^T M \text{ implies } p^T d \geq q^T d]$.

Given a CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$ the activity matrix M is said to be **weakly proper** if for all $p, q \in \mathbb{R}_+^m$ and $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap (\mathbb{R}_+^m \setminus \{0\})$: $[p^T M \geq q^T M \text{ implies } p^T d \geq q^T d]$.

Clearly the activity matrix M is weakly proper if the CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$ is weakly proper.

Since any point in \mathbb{R}^m can always be expressed as the difference between two points in \mathbb{R}_+^m the following is an immediate consequence of the definition of a weakly proper activity matrix.

CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$ is weakly proper for J if and only if for all $p \in \mathbb{R}^m$, any array of non-negative real numbers $\langle \alpha_j | j \in W \cap J \rangle$ and $d \in (CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap (\mathbb{R}_+^m \setminus \{0\}))$: $[p^T M - \alpha_j \geq 0 \text{ for all } j \in W \cap J, p^T M^j \geq 0 \text{ for all } j \in J \setminus W, \text{ implies } p^T d - \sum_{j \in W \cap J} \alpha_j \bar{x}_j \geq 0]$.

Note. The definition corresponding to weakly proper activity matrices in A. Villar (Villar, 2003) is equivalent to our definition of weakly proper activity matrices of CLAAM's because A. Villar (Villar, 2003) requires $\text{Span}(M) \cap \mathbb{R}_+^m \neq \emptyset$.

Lemma 1. Suppose $CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m \neq \emptyset$. Then $(M, \langle \bar{x}_j | j \in W \rangle)$ is a weakly proper CLAAM for J if and only if $\forall p, q \in \mathbb{R}_+^m$, any array of non-negative real numbers $\langle \alpha_j | j \in W \cap J \rangle$ and $d \in CS(M) \cap \mathbb{R}_+^m$: $[p^T M^j - \alpha_j \geq q^T M^j \forall j \in W \cap J \text{ and } p^T M^j \geq q^T M^j \forall j \in W \setminus J \text{ implies } p^T d - \sum_{j \in W \cap J} \alpha_j \bar{x}_j \geq q^T d]$.

Proof. If $(M, \langle \bar{x}_j | j \in W \rangle)$ is a weakly proper CLAAM for J , then it is easy to see that $\forall p, q \in \mathbb{R}_+^m$, any array of non-negative real numbers $\langle \alpha_j | j \in W \cap J \rangle$ and $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m$: $[p^T M^j - \alpha_j \geq q^T M^j \forall j \in W \cap J \text{ and } p^T M^j \geq q^T M^j \forall j \in W \setminus J \text{ implies } p^T d - \sum_{j \in W \cap J} \alpha_j \bar{x}_j \geq q^T d]$.

Hence suppose that $\forall p, q \in \mathbb{R}_+^m$, any array of non-negative real numbers $\langle \alpha_j | j \in W \cap J \rangle$ and $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m$: $[p^T M^j - \alpha_j \geq q^T M^j \forall j \in W \cap J \text{ and } p^T M^j \geq q^T M^j \forall j \in W \setminus J \text{ implies } p^T d - \sum_{j \in W \cap J} \alpha_j \bar{x}_j \geq q^T d]$.

Then as in the case of weakly proper CLAAM's for J , we get in this case that for all $p \in \mathbb{R}^m$, an array of non-negative real numbers $\langle \alpha_j | j \in W \cap J \rangle$ and $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m$: $[p^T M^j - \alpha_j \geq 0 \forall j \in W \cap J, p^T M^j \geq 0 \forall j \in W \setminus J]$, implies $p^T d - \sum_{j \in W \cap J} \alpha_j \bar{x}_j \geq 0$.

Suppose $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap (\mathbb{R}_+^m \setminus \{0\})$.

By hypothesis $CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m \neq \emptyset$. Let $d^* \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m$.

Now $d^* \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m$ implies $td^* \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m$ for all $1 \geq t > 0$.

Similarly, $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m$ implies $(1-t)d + td^* \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m$ for all $1 \geq t > 0$.

Thus, $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m$ implies $(1-t)d + td^* \in CS(M) \cap \mathbb{R}_+^m$ for all $1 \geq t > 0$.

Let $\langle t^{(h)} | h \in \mathbb{N} \rangle$ be a sequence of positive real numbers less than or equal to 1, converging to 0.

Clearly, the sequence $\langle (1-t^{(h)})d + t^{(h)}d^* | h \in \mathbb{N} \rangle$ converges to d and for all $h \in \mathbb{N}$, $p^T((1-t^{(h)})d + t^{(h)}d^*) \geq 0$.

Thus $p^T d \geq 0$.

Thus, $(M, \langle \bar{x}_j | j \in W \rangle)$ is a weakly proper CLAAM for J . ■

Proposition 1. $(M, \langle \bar{x}_j | j \in W \rangle)$ is a weakly proper CLAAM for J if and only if for all final demand vectors $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J)$ there exists $x \in \mathbb{R}_+^n$ satisfying $Mx = d$, $x_j \leq \bar{x}_j$ for $j \in W \cap J$ and $x_j = 0$ for all $j \notin J$.

Proof. Let d be a final demand vector in $CS(M, \langle \bar{x}_j | j \in W \rangle, J)$. For $d = 0$, clearly $Mx = d$, where $x = 0$ and further $x_j \leq \bar{x}_j$ for $j \in W$. Hence, we may suppose that $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m \setminus \{0\}$.

$(M, \langle \bar{x}_j | j \in W \rangle)$ is weakly proper for J if and only if there does not exist $p \in \mathbb{R}^m$, any array of non-negative real numbers $\langle \alpha_j | j \in W \cap J \rangle$ and $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap \mathbb{R}_+^m$ satisfying $p^T M^j - \alpha_j \geq 0 \forall j \in W \cap J$, $p^T M^j \geq 0 \forall j \in W \setminus J$ and $p^T d - \sum_{j \in W \cap J} \alpha_j \bar{x}_j < 0$.

Hence by Farka's lemma, $(M, \langle \bar{x}_j | j \in W \rangle)$ is weakly proper for J if and only if $Mx = d$, $x \in \mathbb{R}_+^n$, $x_j \leq \bar{x}_j$ for $j \in W \cap J$ and $x_j = 0$ for all $j \notin J$ has a solution. ■

Note. Nowhere have we invoked any restriction on the size of the activity matrix or its rank, except that the rank of the activity matrix is positive. That leaves out the uninteresting case of $M = 0$. Thus, our framework is considerably more general than that of Villar (Villar, 2003).

An immediate consequence of the proposition above is that the requirement of $n \leq m$ in Villar (Villar, 2003) can be dispensed with not only for solvability problem in activity analysis, but also for the non-substitution theorem (theorem 5) in the same paper.

5. EXISTENCE OF EQUILIBRIUM PRICE-VECTOR

We will now present a similar generalization as above for the existence of an equilibrium price-vector for a CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$.

Let A_{m+1} be a row vector in \mathbb{R}^n with all co-ordinates strictly positive, where the entry in the column j denoted $a_{m+1,j} > 0$, is the amount of the only non-produced good called "labour" that is used as input if activity j is operated at unit level. Let $\bar{L} > 0$ be the total initial amount of labour in the economy.

Recall that a price-vector is a vector $p \in \mathbb{R}_+^m \setminus \{0\}$. Let $w > 0$ denote wage rate of labour.

At price-vector p and wage rate w the **profit-vector** at the pair (p, w) denoted $\pi(p, w) = p^T M - wA_{m+1}$.

A row vector $v \in \mathbb{R}_+^n$ is said to be **profitable** at wage rate $w > 0$, if there exists $q \in \mathbb{R}^m$ such that $q^T M = wA_{m+1} + v$.

Note. The vector q in the definition of profitable vectors need not be non-negative.

A price-vector p is said to be an **equilibrium price-vector** at the wage rate $w > 0$ and row-vector $v \in \mathbb{R}_+^n$ if $v = \pi(p, w)$.

Recall that an activity matrix M is said to be weakly proper if for all $p, q \in \mathbb{R}_+^m$ and $d \in CS(M, \langle \bar{x}_j | j \in W \rangle, J) \cap (\mathbb{R}_+^m \setminus \{0\})$: $[p^T M \geq q^T M \text{ implies } p^T d \geq q^T d]$.

Proposition 2. Given a CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$, suppose M is a weakly proper activity matrix and v is a profitable row vector at wage rate $w > 0$. Then there exists an equilibrium price-vector p at the wage rate w and row-vector v .

Proof. Since $w > 0$, $A_{m+1} > 0$ and $v \geq 0$, we get $wA_{m+1} + v > 0$.

Since $v \in \mathbb{R}_+^n$ if there exists $p \in \mathbb{R}_+^m$ such that $p^T M = wA_{m+1} + v$, then it must be the case that $p \in \mathbb{R}_+^m \setminus \{0\}$.

Since v is profitable at wage rate $w > 0$, there exists $q \in \mathbb{R}^m$ such that $q^T M = wA_{m+1} + v$.

Towards a contradiction suppose that there does not exist $p \in \mathbb{R}_+^m$ such that $p^T M = wA_{m+1} + v$. By Farkas' lemma, there exists $x \in \mathbb{R}^m$ such that $Mx \in \mathbb{R}_+^m$ and $(wA_{m+1} + v)x < 0$.

Since M is weakly proper, $Mx \in \mathbb{R}_+^m$ implies that there exists $y \in \mathbb{R}_+^n$ such that $My = Mx \geq 0$.

Now $q^T M = wA_{m+1} + v > 0$ and $y \geq 0$ implies $q^T My = (wA_{m+1} + v)y \geq 0$.

Thus $My = Mx$ implies $q^T Mx = q^T My = (wA_{m+1} + v)y \geq 0$.

On the other hand $q^T M = wA_{m+1} + v$ implies $q^T Mx = (wA_{m+1} + v)x < 0$, contradicting $q^T Mx \geq 0$, that we obtained above.

Thus there exists $p \in \mathbb{R}_+^m$ such that $p^T M = wA_{m+1} + v$ and as we observed earlier this $p \in \mathbb{R}_+^m \setminus \{0\}$. ■

6. NON-SUBSTITUTION THEOREM

Recall that a final demand vector is a column vector $d \in \mathbb{R}_+^m \setminus \{0\}$.

Let J be a non-empty subset of $\{1, \dots, n\}$.

Given a CLAAM $(M, \langle \bar{x}_j \mid j \in W \rangle)$, a final demand vector d is said to be **producible by** (activities in) J if there exists $x \in \mathbb{R}_+^n$ satisfying $Mx = d$, $A_{m+1}x \leq \bar{L}$, $x_j \leq \bar{x}_j$ for all $j \in W$, and $x_j = 0$ for all $j \notin J$. Clearly any such x must belong to $\mathbb{R}_+^n \setminus \{0\}$.

It follows from proposition 1, that if $(M, \langle \bar{x}_j \mid j \in W \rangle)$ is weakly proper for J , then any final demand vector $d \in CS(M, \langle \bar{x}_j \mid j \in W \rangle, J)$ is producible by J , provided the requirement of labour to produce it does not exceed \bar{L} .

If $J = \{1, \dots, n\}$, then a final demand vector producible by J is said to be **producible**.

Hence the set of all final demand vectors producible by J is $\{Mx \in \mathbb{R}_+^m \setminus \{0\} \mid x \in \mathbb{R}_+^n, x_j \leq \bar{x}_j \text{ for all } j \in W, A_{m+1}x \leq \bar{L}, \text{ and } x_j = 0 \text{ for all } j \notin J\}$.

Given a price-vector p , a wage rate $w > 0$, a producible final demand vector d , and $x \in \mathbb{R}_+^n$ satisfying $Mx = d$, the aggregate profit of the production sector is $p^T d - wA_{m+1}x$. If the production sector intended to maximize profit, then it would be required to solve the following profit maximization problem: Find x to solve $p^T d - wA_{m+1}x \rightarrow \max$ subject to $Mx = d$, $A_{m+1}x \leq \bar{L}$, $x_j \leq \bar{x}_j$ for all $j \in W$, $x \geq 0$.

However, given the price-vector p and w , the above for a producible final demand vector is equivalent to solving the following linear programming problem denoted $LP - d$:

$$wA_{m+1}x \rightarrow \min \text{ subject to } Mx = d, -A_{m+1}x \geq -\bar{L}, -x_j \geq -\bar{x}_j \text{ for all } j \in W, x \geq 0.$$

If y solves $LP - d$ then $y > 0$ since $d > 0$. Thus $A_{m+1}y > 0$.

The question that we are interested in is the following: If for some producible final demand vector, x^* is an optimal solution for the minimization problem, then is it the case that for all final demand vectors producible by $\{j \mid x_j^* > 0\}$, there exists an optimal solution for the minimization problem, such that the activities operated at a positive level at this optimal solution is a subset of $\{j \mid x_j^* > 0\}$?

Given a producible final demand vector d , the dual of $LP - d$ denoted $DLP - d$ is the following linear programming problem: find $q \in \mathbb{R}^m$, an array of non-negative real numbers $\langle h_j \mid j \in W \rangle$ and a real number $\alpha \geq 0$ to solve:

$$q^T d - \alpha \bar{L} - \sum_{j \in W} h_j \bar{x}_j \rightarrow \max \text{ subject to } q^T M^j - \alpha A_{m+1,j} - h_j \leq wA_{m+1,j} \text{ for all } j \in W, \\ q^T M^j - \alpha A_{m+1,j} \leq wA_{m+1,j} \quad \forall j \notin W.$$

Suppose there exists a producible final demand vector d^* and let x^* be an optimal solution for $LP - d^*$. By the Weak Duality Theorem for LP , there exists $q^* \in \mathbb{R}^m$, an array of non-negative real numbers $\langle h_j^* \mid j \in W \rangle$ and a real number $\alpha^* \geq 0$ such that:

- I) $q^{*T} M^j - \alpha^* A_{m+1,j} - h_j^* \leq wA_{m+1,j}$ for all $j \in W$,
- II) $q^{*T} M^j - \alpha^* A_{m+1,j} \leq wA_{m+1,j}$ for all $j \notin W$,
- III) $[q^{*T} M^j - \alpha^* A_{m+1,j} - h_j^* - wA_{m+1,j}] x_j^* = 0$ for all $j \notin W$,

$$\text{IV)} [q^{*T}M^j - \alpha^*A_{m+1,j} - wA_{m+1,j}]x_j^* = 0 \text{ for all } j \notin W,$$

$$\text{V)} Mx^* = d^*,$$

$$\text{VI)} \alpha^*[A_{m+1}x^* - \bar{L}] = 0,$$

$$\text{VII)} x_j^* \leq \bar{x}_j \quad \forall j \in W,$$

$$\text{VIII)} [x_j^* - \bar{x}_j] = 0 \quad \forall j \notin W,$$

$$\text{IX)} [q^{*T}d^* - \alpha^*\bar{L} - \sum_{j \in W} h_j^* \bar{x}_j] = wA_{m+1}x^*.$$

Since by (V) $Mx^* = d^*$, (III), (IV) and (VI) implies $[q^{*T}d^* - \alpha^*\bar{L} - \sum_{j \in W} h_j^* x_j^*] = wA_{m+1}x^*$.

$[q^{*T}d^* - \alpha^*\bar{L} - \sum_{j \in W} h_j^* x_j^*] = wA_{m+1}x^*$ combined with (VIII) implies $[q^{*T}d^* - \alpha^*\bar{L} - \sum_{j \in W} h_j^* \bar{x}_j] = wA_{m+1}x^*$, which is (IX).

Thus, (III), (IV), (V), (VI) and (VIII) implies (IX).

Hence the required system of equations and inequalities are:

$$\text{I)} q^{*T}M^j - \alpha^*A_{m+1,j} - h_j^* \leq wA_{m+1,j} \text{ for all } j \in W,$$

$$\text{II)} q^{*T}M^j - \alpha^*A_{m+1,j} \leq wA_{m+1,j} \text{ for all } j \notin W,$$

$$\text{III)} [q^{*T}M^j - \alpha^*A_{m+1,j} - h_j^* - wA_{m+1,j}]x_j^* = 0 \text{ for all } j \in W,$$

$$\text{IV)} [q^{*T}M^j - \alpha^*A_{m+1,j} - wA_{m+1,j}]x_j^* = 0 \text{ for all } j \notin W,$$

$$\text{V)} Mx^* = d^*,$$

$$\text{VI)} \alpha^*[A_{m+1}x^* - \bar{L}] = 0,$$

$$\text{VII)} x_j^* \leq \bar{x}_j \text{ for all } j \in W,$$

$$\text{VIII)} [x_j^* - \bar{x}_j]h_j^* = 0 \text{ for all } j \in W.$$

Note that $\{j|x_j^* > 0\} \neq \emptyset$, since $d^* > 0$, $\{j \in W|x_j^* > 0\} \subset \{j|q^{*T}M^j - \alpha^*a_{m+1,j} - h_j^* - wa_{m+1,j} = 0\}$, $\{j \notin W|x_j^* > 0\} \subset \{j \notin W|q^{*T}M^j - \alpha^*a_{m+1,j} - wa_{m+1,j} = 0\}$.

Let $J = \{j|x_j^* > 0\}$.

Let d be a final demand vector producible by J . Then clearly $d \in \{Mx \in \mathbb{R}_+^m \setminus \{0\} | x \in \mathbb{R}_+^n, x_j \leq \bar{x}_j \text{ for all } j \in W, A_{m+1}x \leq \bar{L}, \text{ and } x_j = 0 \text{ for all } j \notin J\}$.

Let $x(d)$ solve the following linear programming problem:

$$wA_{m+1}x \rightarrow \min \text{ subject to } Mx = d, -A_{m+1}x \geq -\bar{L}, -x_j \geq -\bar{x}_j \text{ for all } j \in W, x \geq 0, x_j = 0, \text{ if } j \notin J.$$

Since $J = \{j|x_j^* > 0\} = \{j \in W|x_j^* > 0\} \cup \{j \notin W|x_j^* > 0\} \subset \{j \in W|q^{*T}M^j - \alpha^*a_{m+1,j} - h_j^* - wa_{m+1,j} = 0\} \cup \{j \in W|q^{*T}M^j - \alpha^*a_{m+1,j} - wa_{m+1,j} = 0\}$ and $x_j(d) = 0$ for all $j \notin J$ it is clear that $[q^{*T}M^j - \alpha^*A_{m+1,j} - h_j^* - wA_{m+1,j}]x_j(d) = 0$ for all $j \in W$ and $[q^{*T}M^j - \alpha^*A_{m+1,j} - wA_{m+1,j}]x_j(d) = 0$ for all $j \notin W$.

Hence, the following system of equations and inequalities are satisfied:

$$\text{a)} Mx(d) = d,$$

$$\text{b)} A_{m+1}x(d) \leq \bar{L},$$

$$\text{c)} x_j(d) \leq \bar{x}_j \text{ for all } j \in W,$$

$$\text{d)} q^{*T}M^j - \alpha^*A_{m+1,j} - h_j^* \leq wA_{m+1,j} \text{ for all } j \in W,$$

$$\text{e)} q^{*T}M^j - \alpha^*A_{m+1,j} \leq wA_{m+1,j} \text{ for all } j \notin W,$$

$$\text{f)} [q^{*T}M^j - \alpha^*A_{m+1,j} - h_j^* - wA_{m+1,j}]x_j(d) = 0, \text{ for all } j \in W,$$

$$\text{g)} [q^{*T}M^j - \alpha^*A_{m+1,j} - wA_{m+1,j}]x_j(d) = 0, \text{ for all } j \notin W.$$

Thus $x(d)$ satisfies all the constraints of $LP - d$ and $q^*, \alpha^*, \langle h_j^* | j \in W \rangle$ solves all the constraints of the $DLP - d$. Further, $x_j(d) = 0$, if $j \notin J$.

The value of the objective function of $LP - d$ at $x(d)$ is $wA_{m+1}x(d)$ and that of the dual $DLP - d$ at $q^*, \alpha^*, \langle h_j^* | j \in W \rangle$ is $q^{*T}d - \alpha^*\bar{L} - \sum_{j \in W} h_j^* \bar{x}_j$.

From (a) we get $q^{*T}d = q^{*T}Mx(d)$ and this combined with (f) and (g) gives us $[q^{*T}d - \alpha^*\bar{L} - \sum_{j \in W} h_j^* \bar{x}_j] = wA_{m+1}x(d)$.

At this point we invoke an assumption about activities operating up to “full capacity”.

Assumption (about activities using their entire capacity): suppose that $\{j \in W | x_j^* = \bar{x}_j\} \subset \{j \in W | x_j(d) = \bar{x}_j\}$.

$$\text{Now, } \sum_{j \in W} h_j^* x_j(d) = \sum_{\{j \in W | x_j^* < \bar{x}_j\}} h_j^* x_j(d) + \sum_{\{j \in W | x_j^* = \bar{x}_j\}} h_j^* x_j(d).$$

$$\text{Clearly } \sum_{\{j \in W | x_j^* < \bar{x}_j\}} h_j^* x_j(d) = 0, \text{ since } h_j^* = 0 \text{ whenever } x_j^* < \bar{x}_j.$$

$$\text{Thus, } \sum_{j \in W} h_j^* x_j(d) = \sum_{\{j \in W | x_j^* = \bar{x}_j\}} h_j^* x_j(d).$$

$$\text{However, } \{j \in W | x_j^* = \bar{x}_j\} \subset \{j \in W | x_j(d) = \bar{x}_j\}.$$

$$\text{Thus, } \sum_{j \in W} h_j^* x_j(d) = \sum_{\{j \in W | x_j^* = \bar{x}_j\}} h_j^* \bar{x}_j.$$

$$\text{Since } \sum_{\{j \in W | x_j^* < \bar{x}_j\}} h_j^* \bar{x}_j = 0, \text{ we get } \sum_{j \in W} h_j^* \bar{x}_j = \sum_{j \in W} h_j^* x_j(d).$$

$$\text{We already have, } [q^{*T}d - \alpha^* \bar{L} - \sum_{j \in W} h_j^* x_j(d)] = wA_{m+1}x(d).$$

$$\text{Substituting } \sum_{j \in W} h_j^* \bar{x}_j \text{ for } \sum_{j \in W} h_j^* x_j(d) \text{ in the above equation gives } [q^{*T}d - \alpha^* \bar{L} - \sum_{j \in W} h_j^* \bar{x}_j] = wA_{m+1}x(d).$$

Thus, as is well known in the theory of linear programming, $x(d)$ is an optimal solution for $LP - d$ and so the answer to the question we have posed earlier is in the affirmative, **provided** $\{j \in W | x_j^* = \bar{x}_j\} \subset \{j \in W | x_j(d) = \bar{x}_j\}$.

From (d) and (e) we get $q^{*T}M^j - \alpha^*A_{m+1,j} - h_j^* \leq wA_{m+1,j}$ for all $j \in W$ and $q^{*T}M^j - \alpha^*A_{m+1,j} \leq wA_{m+1,j}$ for all $j \notin W$.

Thus for all $j \in \{1, \dots, n\}$, there exists a non-negative real number ε_j such that $q^{*T}M^j - wA_{m+1,j} = \alpha^*A_{m+1,j} + \varepsilon_j$.

Clearly, $\varepsilon_j = h_j^*$ for all $j \in W$ satisfying $q^{*T}M^j - \alpha^*A_{m+1,j} - h_j^* - wA_{m+1,j} = 0$ and $\varepsilon_j = 0$ for all $j \notin W$ satisfying $q^{*T}M^j - \alpha^*A_{m+1,j} - wA_{m+1,j} = 0$.

Let $v \in \mathbb{R}_+^n$ be the row-vector whose coordinate j is $\alpha^*A_{m+1,j} + \varepsilon_j$.

Since $q^{*T}M - wA_{m+1} = v$, v is profitable at wage rate w .

Hence if M is a weakly productive activity matrix, by Proposition 2 it follows that there exists an equilibrium price-vector p^* at the wage rate w and row-vector v .

Important Note: v depends on $q^*, \alpha^*, \langle h_j^* | j \in W \rangle$ which depends on J . Thus p^* depends on the producibility of d by activities in J and on the assumption $\{j \in W | x_j^* = \bar{x}_j\} \subset \{j \in W | x_j(d) = \bar{x}_j\}$.

Hence, as mentioned in the first section, there is a clear **dependence** of the **equilibrium price-vector** on the **final demand vector**, unlike the conclusion of Sraffian economics.

This proves the following theorem, which is popularly known as the Non-Substitution Theorem.

Theorem 1. Given a CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$, suppose that for some producible final demand vector d^*, x^* is an optimal solution for $LP - d^*$. Let d be a final demand vector producible by $J = \{j | x_j^* > 0\}$. Let $x(d)$ be an optimal solution for the linear programming problem $LP - d$ along with an additional constraint $x_j = 0$ for all $j \notin J$ (i.e., x is producible by J):

$$wA_{m+1}x \rightarrow \min \quad \text{s.t. } Mx = d^*, A_{m+1}x \leq \bar{L}, x_j \leq x_j^* \quad \forall j \in W, x_j = 0 \quad \forall j \notin J, x \geq 0.$$

If $\{j \in W | x_j^* = \bar{x}_j\} \subset \{j \in W | x_j(d) = \bar{x}_j\}$ (i.e., the capacities that are binding at x^* continue to remain binding at $x(d)$), **then** $x(d)$ solves $LP - d$.

If in addition M is weakly productive, then there exists a price-vector p^* known as efficiency price-vector, an array of non-negative real numbers $\langle h_j^* | j \in W \rangle$ and a real number $\alpha^* \geq 0$ — such that:

- i) $p^T M^j - wA_{m+1,j} - h_j^* \leq \alpha^* A_{m+1,j}$ for all $j \in W$;
- ii) $p^T M^j - wA_{m+1,j} \leq \alpha^* A_{m+1,j}$ for all $j \notin W$;
- iii) $p^T M^j - wA_{m+1,j} - h_j^* = \alpha^* A_{m+1,j}$ for all $j \in W$ with $x_j^* > 0$;
- iv) $p^T M^j - wA_{m+1,j} = \alpha^* A_{m+1,j}$ for all $j \notin W$ with $x_j^* > 0$. ■

Note. In the statement above, for each $j \in W$, h_j^* could be interpreted as the shadow price of operating the activity j at unit level. So, the shadow price of any activity that operates below capacity at x^* is 0, and continues to remain, so even if at $x(d)$ it uses up the entire capacity.

An immediate corollary of Theorem 1 is the following “compact” result valid only for (M, φ) .

Corollary of Theorem 1. Given a CLAAM (M, φ) , suppose that for some producible final demand vector d^* , x^* is an (a basic) optimal solution for $LP - d^*$. Let d be a final demand vector producible by $J = \{j | x_j^* > 0\}$. Let $x(d)$ be an optimal solution for $LP - d$ along with an additional constraint “ x is producible by J ”. Then $x(d)$ solves $LP - d$.

If in addition M is weakly productive, then there exists a price-vector p^* known as efficiency price-vector and a real number $\alpha^* \geq 0$ — such that:

- i) $p^T M^j - wA_{m+1,j} \leq \alpha^* A_{m+1,j}$ for all $j \in \{1, \dots, n\}$;
- ii) $p^T M^j - wA_{m+1,j} = \alpha^* A_{m+1,j}$ for all j with $x_j^* > 0$.

Note. In the proof of Theorem 1 presented in the form of a discussion prior to the statements of the two theorems, observe that, since $x(d)$ must belong to $\mathbb{R}_+^n \setminus \{0\}$ and $A_{m+1} \gg 0$, it must be the case that $wA_{m+1}x(d) > 0$.

7. MULTISECTOR PRODUCTION THEORY FOR CLAAM

The implications of the above analysis for constrained linear activity analysis models when the m manufactured goods are interpreted as m distinct composite commodities in a one-to-one correspondence with m distinct sectors of the economy is best performed with a dynamic (two-period) interpretation of the model discussed here, with production taking place during the current/first period — period 0 and the final demand vector is supplied during a subsequent period — period 1. In such a situation a CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$ corresponds to the specific case where:

- a) there exist non-negative $m \times n$ matrices B, A such that $M = B - A$;
- b) the entries in the matrices B and A are measured in money units evaluated at producer prices.

Thus for $I \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, a_{ij} is the cost of good i required to operate activity j at unit level and b_{ij} is the monetary of good i produced if activity j is operated at unit level, both measured in producer prices prevailing in period 0.

Issues relating to solvability remain intact — the analysis in section four remains unaffected under this new interpretation. What however requires some reformulations are issues related to existence of equilibrium price vector and as a consequence, issues related to efficiency prices via the non-substitution theorems.

In the first place, instead of price-vector the concept that is relevant in this context is (sectoral) inflation rate vectors, i.e., the vector of factors by which the period 0 sectoral price indices are individually multiplied to obtain the period 1 price indices.

As in section 5, the amount of the only non-produced good called labour that is used as input as well its total initial amount in the economy is measured in physical units.

An **inflation rate-vector** is a vector $p \in \mathbb{R}_+^m \setminus \{0\}$. Let $w > 0$ denote wage rate of labour.

At inflation rate-vector p and wage rate w the **profit-vector** at the pair (p, w) denoted $p(p, w) = p^T B - e^T A - wA_{m+1}$, where e is the m -dimensional column vector all entries of which are 1; e is called the sum-vector.

A row-vector $v \in \mathbb{R}_+^n$ is said to be **profitable** at wage rate $w > 0$, if there exists $q \in \mathbb{R}^m$ such that $q^T B = wA_{m+1} + e^T A + v$.

Note. The vector q in the definition of profitable vectors need not be non-negative.

An inflation rate-vector p is said to be an **equilibrium inflation rate-vector** at the wage rate $w > 0$ and row-vector $v \in \mathbb{R}_+^n$ if $v = \pi(p, w)$.

Instead of the activity matrix being weakly proper, we now require the following property.

Weakly proper Output Coefficient Matrix. Given a CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$ with $M = B - A$ for non-negative matrices A and B , for all $p, q \in \mathbb{R}_+^m$ and $d \in CS(M, \langle \bar{x}_j | j \in W \rangle) \cap (\mathbb{R}_+^m \setminus \{0\})$: $[p^T B \geq q^T B$ implies $p^T d \geq q^T d]$.

This allows us to state and prove the following result.

Result 1. Given a CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$ with $M = B - A$ for non-negative matrices A and B and satisfying weakly proper output coefficient matrix property if v is a profitable row vector at wage rate $w > 0$. Then there exists an equilibrium inflation rate-vector p at the wage rate w and row-vector v .

Note that the solvability aspect of the non-substitution theorem, concerns minimization of aggregate wages or cost of labour, subject to producibility constraint. Hence that aspect of the non-substitution theorem remains intact under the new interpretation. We need to however replace the concept of efficiency prices by efficiency inflation rates.

The following result has a proof similar to the one for theorem 1.

Result 2. Given a CLAAM $(M, \langle \bar{x}_j | j \in W \rangle)$ with $M = B - A$ for non-negative matrices A and B , suppose that for some producible final demand vector d^* , x^* is an optimal solution for $LP - d^*$. Let d be a final demand vector producible by $J = \{j | x_j^* > 0\}$. Let $x(d)$ be an optimal solution for the linear programming problem $LP - d$ **along with an additional constraint** $x_j = 0$ for all $j \notin J$ (i.e., x is producible by J):

$$wA_{m+1}x \rightarrow \min \quad s.t. \quad Mx = d^*, \quad A_{m+1}x \leq \bar{L}, \quad x_j \leq \bar{x}_j \quad \forall j \in W, \quad x_j = 0 \quad \forall j \notin J, \quad x \geq 0.$$

If $\{j \in W | x_j^* = \bar{x}_j\} \subset \{j \in W | x_j(d) = \bar{x}_j\}$ (i.e., the capacities that are binding at x^* continue to remain binding at $x(d)$), **then** $x(d)$ solves $LP - d$.

If in addition the Weakly Proper Output Coefficient Matrix property is satisfied, then there exists an inflation rate-vector p^* known as efficiency inflation rate vector, an array of non-negative real numbers $\langle h_j^* | j \in W \rangle$ and a real number $\alpha^* \geq 0$ — such that:

- i) $p^T B^j - e^T A^j - wA_{m+1,j} - h_j^* \leq \alpha^* A_{m+1,j}$ for all $j \in W$;
- ii) $p^T B^j - e^T A^j - wA_{m+1,j} \leq \alpha^* A_{m+1,j}$ for all $j \notin W$;
- iii) $p^T B^j - e^T A^j - wA_{m+1,j} - h_j^* = \alpha^* A_{m+1,j}$ for all $j \in W$ with $x_j^* > 0$;
- iv) $p^T B^j - e^T A^j - wA_{m+1,j} = \alpha^* A_{m+1,j}$ for all $j \notin W$ with $x_j^* > 0$.

An immediate corollary of Result 2 is the following compact result valid only for (M, ϕ) .

So, the shadow price of any activity that operates below capacity at x^* is 0, and continues to remain, so even if at $x(d)$ it uses up the entire capacity.

Corollary of Result 2. Given a CLAAM (M, ϕ) , suppose that for some producible final demand vector d^* , x^* is an (a basic) optimal solution for $LP - d^*$. Let d be a final demand vector producible by $J = \{j | x_j^* > 0\}$. Let $x(d)$ be an optimal solution for $LP - d$ **along with an additional constraint** x is producible by J . Then $x(d)$ solves $LP - d$.

If in addition Weakly Proper Output Coefficient Matrix property is satisfied, then there exists an inflation rate-vector p^* known as **efficiency inflation rate-vector** and a real number $\alpha^* \geq 0$ — such that:

- i) $p^T B^j - e^T A^j - wA_{m+1,j} \leq \alpha^* A_{m+1,j}$ for all $j \in \{1, \dots, n\}$;
- ii) $p^T B^j - e^T A^j - wA_{m+1,j} = \alpha^* A_{m+1,j}$ for all j with $x_j^* > 0$.

In the case of final demand for services (provided by the service sector) which do not require manufacturing, but are none the less measured in current producer prices, this may imply a difference between producer prices and the prices that consumers are required to pay for the services during the current period.

REFERENCES / СПИСОК ЛИТЕРАТУРЫ

- Chander P.** (1983). The nonlinear input-output model. *Journal of Economic Theory*, 30, 219–229.
- Lahiri S.** (2022). *The essential appendix on linear programming*. Available at: https://www.academia.edu/44541645/The_essential_appendix_on_Linear_Programming
- Lancaster K.** (1968). *Mathematical economics*. New York: The Macmillan Company.
- Sandberg I.W.** (1973). A nonlinear input-output model of a multisector economy. *Econometrica*, 41, 6, 1167–1182.
- Villar A.** (2003). The generalized linear production model: Solvability, non-substitution and productivity measurement. *Advances in Theoretical Economics*, 3, 1, 1.

Acknowledgment. This paper was written when I was a Professor at the School of Petroleum Management, PD Energy University, from where I retired on June 5, 2022. An earlier version of this paper was presented (virtually) at a seminar on April 5, 2020, in the department of Industrial Engineering and Operations Research, IIT-Bombay. I would like to thank K.S. Mallikarjuna Rao for observations about the paper during my presentation. A subsequent version of this paper was presented (virtually) at the 9th International Conference on Matrix Analysis and Applications hosted by University of Aveiro, Portugal from June 15–17, 2022. I wish to the conference participants — in particular Enide Aandrade — for a positive assessment of the paper. I wish to put on record my deep gratitude to Arabinda Tripathy, for appreciative comments and endorsement of the contents, in addition to encouragement for this research — several steps beyond his role as a senior colleague. I gladly and happily put on record “a very big thank you” to my friend Ishan Mistry, whose eyes noticed the innumerable errors due to me and a few that occurred while preparing the final printed version. I would also like to thank an anonymous referee of this journal and Victor Dementiev (Editor-in-Chief) for suggestions towards improvement of this paper. Last but not the least, I wish to express the immense honour I feel in being able to publish this paper in an academically respected journal published in Russia — the birth place of Professor Wassily Leontief — the father of input-output macroeconomics and Professor Leonid Kantorovich — the father of Linear Programming — whose seminal contributions to “applicable mathematical economics” are the ancestors of the work presented in this paper.

Теория производства для содержащих ограничения линейных моделей операций³

© 2023 S. Lahiri

S. Lahiri,

Lok Jagruti (LJ) University, India; e-mail: somdeb.lahiri@gmail.com

Поступила в редакцию 24.09.2022

Аннотация. Цель статьи — расширить рамки анализа операций, которые обсуждались в статье Антонио Виллара без каких-либо требований к размеру матриц. В его работе была введена модель анализа операций, в которой операции могут иметь (или не иметь) ограничений по мощности. Мы применяем обычные приемы анализа затраты—выпуск при определении количества товара внешним для сектора производства (или обрабатывающей промышленности) потребителям и рассматриваем его как конечный спрос. Мы получили сходные с А. Вилларом результаты об алгоритмической разрешимости, незамещаемости и существовании эффективных цен. Мы применили наш анализ и результаты к двухпериодной модели анализа многосекторальной деятельности с ограничениями по мощности. Матрица операций представляет собой разницу между неотрицательной матрицей коэффициентов выпуска и неотрицательной матрицей коэффициентов затрат, при том что коэффициенты были определены в денежных единицах для каждого вида деятельности. Почти все результаты, полученные к настоящему времени, представлены в таком макроэкономическом контексте. Тем не менее, необходимы некоторые изменения формулировок для проблем, относящихся к существованию равновесного вектора цен и — как следствие — к вопросам, относящимся к эффективности цен через теоремы о незамещаемости. Соответствующие концепции применяются здесь и к векторам уровней инфляции.

Ключевые слова: линейный анализ процессов с ограничениями, разрешимость, теорема о незамещаемости, эффективные цены.

Классификация JEL: C67, E16, E23, E31.

Для цитирования: **Lahiri S.** (2023). Production theory for constrained linear activity models // *Экономика и математические методы*. Т. 59. № 1. С. 5–15. DOI: 10.31857/S042473880024866-1

³ В работе сообщаются новые результаты, обобщаются и расширяются границы анализа, приведенного автором в ранней работе «Анализ производства, основывающийся на матричном представлении операций», доступной на <https://drive.google.com/file/d/15E2SIb5NeWBgtRI-O1gb1UA9CG6QhJc/view>