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ТЕОРЕТИЧЕСКИЕ И МЕТОДОЛОГИЧЕСКИЕ ПРОБЛЕМЫ

Recovering the actual trajectory of economic cycles

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Abstract. The paper deals with the development of a method for restoring the trajectory of economic cycles from estimates of the gross domestic product (GDP). The proposed approach to solve this problem is based on the interpretation of cycles in the form of random oscillations of the income with a certain natural frequency, also called a narrowband random process. The operators (Fourier transforms, filtering, etc.) used to recover the cycle trajectory are linear. Their inherent associativity property allows changing the sequence of implementation of the linear operators above. As a result, it is proposed to start the recovery with bandpass filtering of the GDP function, and after that to parry the influence of the inertia property of the GDP estimator. Taking the qualities of a narrowband random process into consideration made it possible to create a simplified procedure to recover the cycle trajectory. In the example of the Kuznets swing, the acceptability of this procedure is demonstrated for the practical econometrics. The developed method is applicable in problems that require knowledge of the trajectory of the considered cycle.

Keywords: economic cycle, random oscillations, cycle trajectory, Fourier transforms, frequency response characteristics.

JEL Classification: C02, C15, C22.

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1. INTRODUCTION

The model of economic cycles, as proposed in reference (Karmalita, 2020), is based on a probabilistic description of the investment function and the perception of the economic system as a material object with certain inherent properties. As for the system model itself, it represents the economic cycle as random oscillations generated by a linear elastic system with the natural frequency $f_0 = 1/T_0$ and a damping coefficient h under the influence of the Gaussian white noise $E(t)$ (Bolotin, 1984). Mathematically, the cycle model is represented by an ordinary differential equation of the second order:

$$\ddot{\Xi}(t) + 2h\dot{\Xi}(t) + (2\pi f_0)^2 \Xi(t) = E(t),$$

where $\Xi(t)$ represents random oscillations (cycle under consideration) of the income function $X(t)$, and $E(t)$ — fluctuations of investments. The values of random oscillations $\Xi(t)$ are determined mainly by harmonics in the frequency band $f_1 = 0.7f_0 \le f \le f_2 = 1.4 f_0$. This fact explains another name for random oscillations — a narrowband random process.

In econometric studies, to quantify the income function $X(t)$ the gross domestic product (GDP), hereinafter $G(t)$, is usually used. Recall that the value of GDP is a monetary estimate of manufactured goods and provisioned services for a certain period ΔT. In the above reference, $G(t)$ is mathematically described by the convolution equation:

$$G(t) = \int_{t-\Delta T}^{t} X(t)\,dt = \int_0^t h(\tau) X(t-\tau)\,d\tau. \quad (1)$$

In other words, the GDP function can be interpreted as the result of measuring the income function using an estimating means (estimator) whose inertial properties are described by the impulse response (IR) function $h(\tau)$:

$$h(\tau) = \begin{cases} 1, & 0 \le \tau \le \Delta T; \\ 0, & \tau < 0;\ \tau > \Delta T. \end{cases}$$

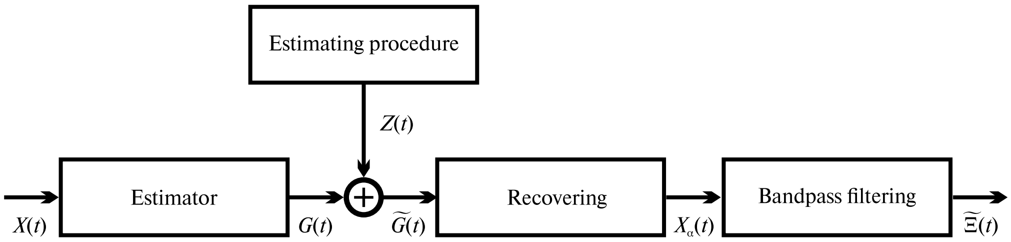

Fig. 1. Forming the cycle trajectory

The reference above shows that the concept of the frequency (period) of the income function is quite applicable to the analysis of economic systems. If we consider equation (1) in the frequency domain, then it is transformed into the multiplication of the corresponding Fourier images:

$$G(f) = H(f)X(f), \quad (2)$$

where $G(f)$, $H(f)$, and $X(f)$ are the Fourier images of the GDP, impulse response and income functions, respectively. Therefore, knowledge of $h(t)$ and $G(t)$ provides an opportunity to solve equation (2) with respect to $X(f) = G(f)/H(f)$. The author (Karmalita, 2020) demonstrates that this problem is ill-posed and offers an approximate solution $X_\alpha(t)$ using the regularization method (Tikhonov, Arsenin, 1997). After recovering $X_\alpha(t)$, the trajectory of the concerned cycle $\Xi(t)$ with a natural frequency f_0 is formed by a bandpass filter (Schlichtharle, 2011).

Let us turn to the recovery procedure with a sample diagram represented in Fig. 1.

This diagram shows the error $Z(t)$ describing the imperfect performance of the estimation procedure. This error arises due to unreliable statistical data, its incompleteness, the performer's skill, etc. Therefore, the available data related to the values of GDP are the estimates $\tilde{G}(t)$ determined as $\tilde{G}(t) = G(t) + Z(t)$.

The function $Z(t)$ is called the measurement (instrumental) noise, which is usually wideband and is often modeled as "white" noise.

The above approach has some features that complicate its application in econometric practice. In particular, the solution of the ill-posed problem of recovering the income function requires a few subjective decisions. The latter include, for example, the assumption about the nature of the smoothness of an unknown multicomponent income function. Another informal decision relates to establishing the range of the regularization parameter α, in which its optimum is sought. The absence of formal justifications of the above decisions requires a certain skill (a kind of art) to successfully solve the ill-posed problem. The specified influence of subjective factors on the recovery procedure determines the purpose of this article — the development of a formal method to obtain the trajectory of the cycle of interest while avoiding an ill-posed formulation of the problem. This method is presented in the second section of the article. Its third section illustrates the developed method with the example of recovering the Kuznets swing.

2. FORMING CYCLE VALUES

In the frequency domain, the estimator is characterized by its frequency response (*FR*) function, denoted $H(f)$, which is the Fourier transform of the *IR* function (Pavleino, Romadanov, 2007):

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft}\, dt = \Delta t \,\mathrm{sinc}(\pi \Delta T f) e^{-j\pi\Delta T f} = A_h(f) e^{j\Phi_h(f)}, \quad (3)$$

where $A_h(f)$ and $\Phi_h(f)$ are the amplitude and phase frequency (*AF* and *PhF*) characteristics of the estimator, while

$$\mathrm{sinc}(\pi\Delta tf) == \begin{cases} 1, & f = 0; \\ (\sin \pi\Delta Tf)/\pi\Delta Tf, & f \neq 0. \end{cases}$$

Views of the above-mentioned *AF* and *PhF* characteristics are shown in Fig. 2.

$A_h(f)$ determines the ratio of the amplitudes of the input and output harmonics of the estimator. $\Phi_h(f)$ is the difference between their phases, which is equivalent to the time delay of the output process with respect to the input process. It follows from the expression (2) that $H(f)$ only change the amplitude and phase of the harmonics

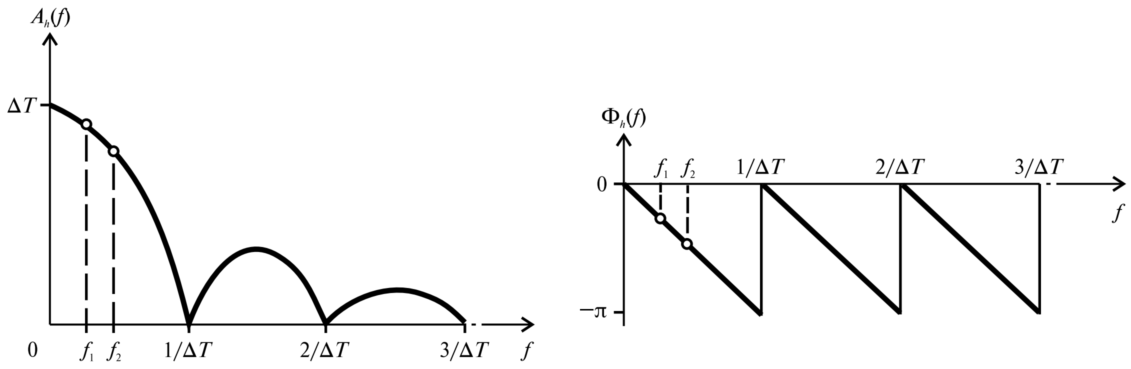

Fig. 2. The frequency characteristics of the estimator

that make up $X(t)$. Despite this, the spectral composition of GDP and income functions is similar. Recall that the intensity (amplitude) of the income's oscillations with the natural frequency f_0 is determined by the frequency range in the vicinity of f_0. This fact led to the use of the bandpass filtering to extract the data of the cycle of interest from the recovered income function. Such filtering allows signals to pass unaltered within the frequency range $f_1 \le f \le f_2$ while suppressing all the others. Amplitude spectrum $A_c(f)$ of the income random oscillations and *AF* characteristics $A_f(f)$ of the bandpass filter are shown in Fig. 3.

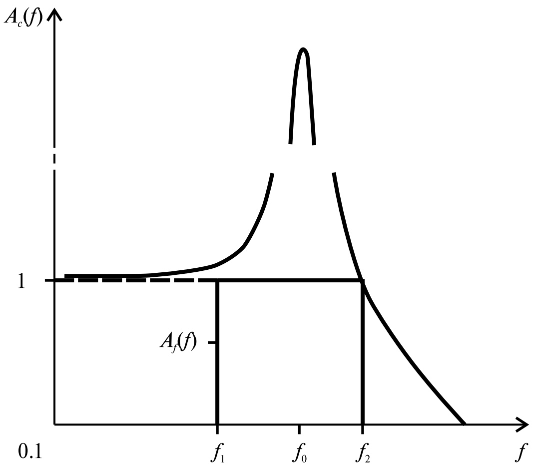

Fig. 3. Amplitude spectrum of the cycle oscillations and *AF* characteristics of the bandpass filter

In principle, extracting some cycle $\Xi(t)$ with natural frequency f_0 does not require a complete recovery of the income function. The associativity inherent in linear convolution and filtering operators allows changing the sequence of operations. Namely, one can first form the GDP function corresponding to the considered cycle, and then recover only the values of the latter (Fig. 4).

It should be noted that in addition to the extract function, this filtering will suppress the noise $Z(t)$ that accompanies the realization $\tilde{G}(t)$.

The presence of filtered realization $G_f(t)$ makes it possible to recover the values of income oscillations in the form of estimates $\tilde{\Xi}(t)$. Using expression (2), the Fourier image $\tilde{\Xi}(t)$ can be represented in terms of *AF* and *PhF* characteristics in the following form:

$$\tilde{\Xi}(f) = G_f(f)/H(f) = \frac{A_{G_f}(f)\mathrm{e}^{j\Phi_{G_f}(f)}}{\vec{A}_h(f)\mathrm{e}^{j\vec{\Phi}_h(f)}} = \frac{A_{G_f}(f)}{\vec{A}_h(f)}\mathrm{e}^{j[\Phi_{G_f}(f)-\vec{\Phi}_h(f)]}.$$

Here $\vec{A}_h(f)$ and $\vec{\Phi}_h(f)$ include the fragments of corresponding *AF* and *PhF* characteristics in the range $f_1 \le f \le f_2$ (see Fig. 3), having the following values:

$$\vec{A}_h(f) = \begin{cases} A_h(f_1), & f < f_1; \\ A_h(f), & f_1 \le f \le f_2; \\ A_h(f_2), & f > f_2, \end{cases} \quad \vec{\Phi}_h(f) = \begin{cases} \Phi_h(f_1), & f < f_1; \\ \Phi_h(f), & f_1 \le f \le f_2; \\ \Phi_h(f_2), & f > f_2. \end{cases}$$

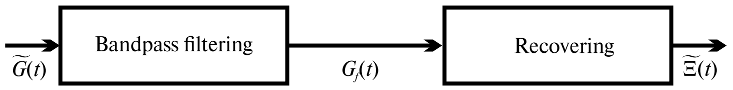

Fig. 4. Proposed recovery approach

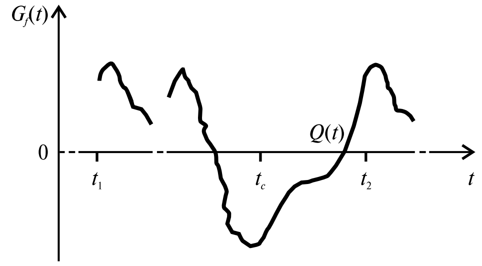

Fig. 5. Fragment of filtered realization $G_f(t)$

The inverse Fourier transform of $\tilde{\Xi}(f)$ will provide the actual trajectory of the considered cycle:

$$\tilde{\Xi}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Xi}(f)\mathrm{e}^{j2\pi ft}\,df.$$

The reconstruction of the time domain processes from its Fourier images has its own specifics. As a rule, the initial and final values of the filtered GDP realization presented on the time interval $t_1, ..., t_c$ are different. The Fourier transform of $G_f(t)$ assumes the periodic nature of this fragment. The specified inequality $G_f(t_1) \neq G_f(t_c)$ leads to the appearance of a gap (discontinuity) between the last value $G_f(t_c)$ and the first one of the realization $G_f(t)$ with its hypothetical continuation. Therefore, the values of the recovered $\tilde{\Xi}(t)$ will oscillate in the regions adjacent to break points (t_1 and t_c) of the recovered trajectory. In (Karmalita, 2020) the following technique is proposed to eliminate the gap in the periodicity of the realization $G_f(t)$. Its endpoint is complemented by a cubic parabola $Q(t)$ in the time interval $t_c, ..., t_2$, as shown in Fig. 5.

$Q(t)$ is a polynomial of the form: $Q(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3$. Its coefficients q_l ($l = 0, ..., 3$) are completely determined by the following boundary requirements:

$$Q(t_c) = G_f(t_c);\ Q(t_2) = G_f(t_1);$$

$$\left.\frac{dQ(t)}{dt}\right|_{t_c} = \left.\frac{dG_f(t)}{dt}\right|_{t_c};\ \left.\frac{dQ(t)}{dt}\right|_{t_2} = \left.\frac{dG_f(t)}{dt}\right|_{t_1}.$$

Such a "loopback" of realization of $G_f(t)$ practically excludes the Gibbs' phenomenon under the inverse Fourier transform.

Let us recall that accepted model of economic cycles considers them to be the processes which values are determined by harmonics in the vicinity of the natural frequency f_0. This fact allows the simplification of the above recovery procedure based on considering the influence of the estimator only on the following harmonic — $\xi(t) = A\sin(2\pi f_0 t)$. The influence of the estimator lies in its transformation to the following form:

$$g(t) = A_h(f_0)\,A\sin[2\pi f_0\,t + \Phi_h(f_0)] = A_h(f_0)\,A\sin(2\pi f_0 t - \pi f_0\,\Delta T) =$$
$$= A_h(f_0)\,A\sin[2\pi f_0(t - \Delta T/2)] = A_h(f_0)\,\xi(t - \Delta T/2).$$

In other words, the estimator changes the amplitude of the harmonic and shifts it in time towards the delay. This fact simplifies the procedure for recovering the business cycle, since only bandpass filtering of the GDP function can provide estimates of the cycle trajectory with an accuracy acceptable for practical econometrics.

3. RECOVERING A FRAGMENT OF THE KUZINETS SWING

Now consider the approach above in the case of recovering the Kuznets swing from quarterly GDP estimates of the USA economy. Such estimates for the time period 2000–2021 are shown in Fig. 6 (Federal Reserve Economic Data, 2022).

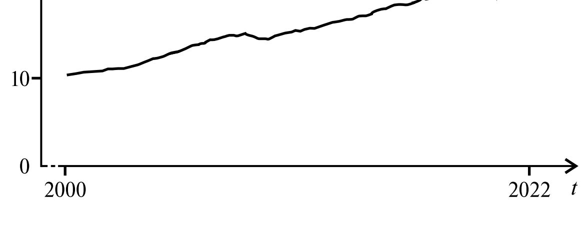

Fig. 6. Real *GDP* estimates of the US economy

In the case of quarterly GDP estimates, the sampling interval is $\Delta t = \Delta T \approx 0.25$ year. The non-equidistance of $G_i = G(i\Delta t)$ readings is related to the different duration of quarters during the year. For the first quarter, Δt_1 is equal to 90 or 91 (leap year) days, $\Delta t_2 = 91$, $\Delta t_3 = 92$, and $\Delta t_4 = 92$. Hence, the additional contribution to the measurement noise $Z(t)$ due to fluctuations in the sampling interval yield an additional randomization of considered processes. Such a randomization leads to a bias (increase) in the estimates of the damping coefficient h. For example, it transforms a harmonic process ($h = 0$) into a narrowband random process.

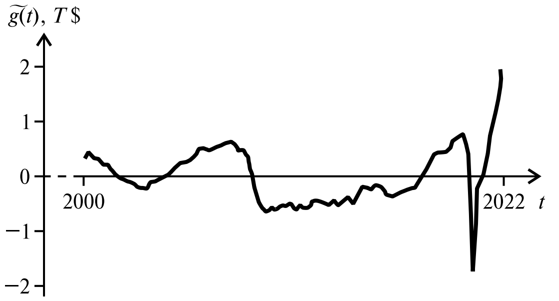

Fig. 7. Fragment of *GDP* oscillations

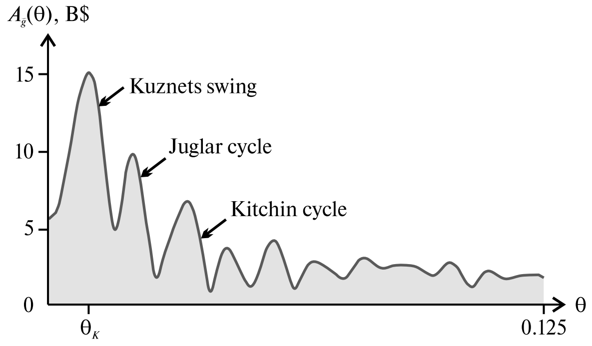

Fig. 8. Spectrogram of *GDP* oscillations

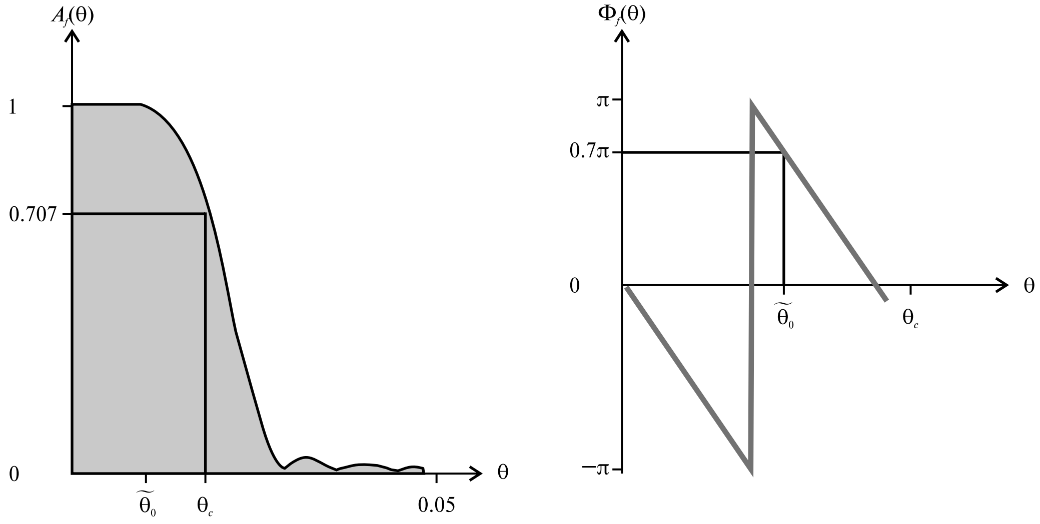

Fig. 9. The *AF* and *PhF* characteristics of the *LP* filter

To increase the representativeness of the empirical data (number of samples), additional terms were calculated via linear interpolation of quarterly GDP estimates. Thus, the sampling interval was reduced to a value of 0.125 years, and the samples number increased to $n = 175$. Subsequently, oscillations $\tilde{g}_i$ of GDP estimates (Fig. 7) were evaluated relative to the regression $\tilde{G}_i = 71.242i + 9609.753$, obtained by the least squares method (Brandt, 2014).

If we proceed to the dimensionless interval of sampling $\Delta t = 1$, then it will correspond to the range of the relative natural frequency $0 \le \theta_0 \le 0.5$. The value of θ_0 of the cycle of interest can be determined through Fourier analysis (Cho, 2018) of oscillations $\tilde{g}_i$ (Fig. 8).

Spectral analysis of oscillations $\tilde{g}_i$ for the period 1871–2007 (Korotaev, Tsirel, 2010) determined that the duration of the Kondratiev cycle is 52–53 years. In that instance, the Kuznets swing was interpreted there as the third harmonic of the Kondratiev cycle with a relative natural frequency $\theta_0 \approx 3/(52{\times}8) \approx 0.0072$. From the above spectrogram the estimate $\tilde{\theta}_K \approx 0.01$.

The low pass (*LP*) filtering of the Kuznets swing may be performed by the *FIR* filter $\tilde{g}_{Ki} = \sum_{l=1}^{128} c_l \tilde{g}_{i+l-128} \left(i = 128,\ldots,175\right)$ with the cut off frequency $\tilde{\theta}_0 = 0.018$ (Fig. 9).

It follows from the figure above that the amplitude of oscillations $S\left(t\right) = 2\pi \tilde{f}_0 t(\tilde{f}_K = 0.125\tilde{\theta}_K)$ does not change in the bandpass of the filter. However, the phase of the filtered process $S_f(t)$ modified as shown in Fig.10.

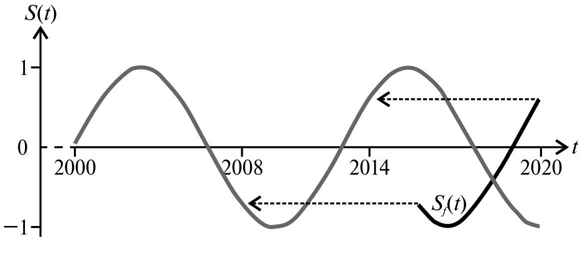

Fig. 10. Effect of filtering a harmonic process

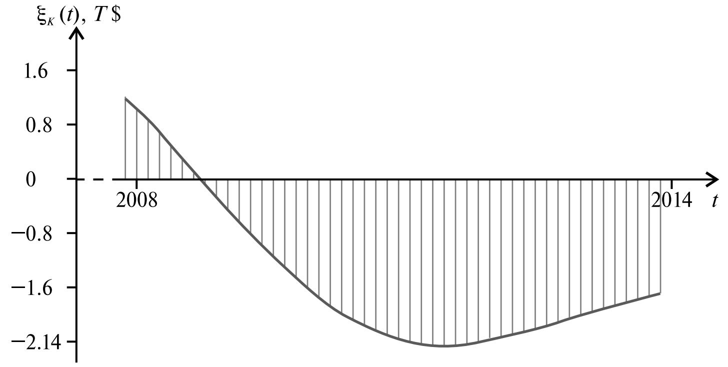

Fig. 11. Estimates of the Kuznets swing trajectory

Due to quarterly ($\Delta T \approx 0.25$) estimates G_i, the value $A_h(\tilde{\theta}_0) = 0.25\sin(0.01\pi)/0.01\pi \approx 0.25$. Thus, the recovered trajectory of the Kuznets swing is $\tilde{\xi}_K(t - 0.125) \approx 4\tilde{g}_K(t)$, and in the case of a discrete representation of data is reduced to the form $\tilde{\xi}_{K(i-1)} \approx 4\tilde{g}_{Ki}$ (Fig. 11).

4. CONCLUSION

A method for recovering the trajectory of the economic cycle, represented by the GDP function, was developed. It is based on the interpretation of cycles in the form of random oscillations in income with a certain natural frequency, which is also referred to as a narrowband random process. The method consists of bandpass filtering of the GDP function and subsequently parrying the influence of the estimator in the passband of the filter. The peculiarities of narrowband processes made it possible to create a simplified procedure for recovering the cycle trajectory. In the example of the Kuznets swing, the feasibility of this approach is demonstrated.

The developed method can be used in the problems where the solution requires knowledge of the trajectory of the cycle under consideration. An example of such a problem is predicting the trajectory of a cycle (Karmalita, 2022).

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Восстановление фактической траектории экономических циклов

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Аннотация. Работа посвящена разработке метода восстановления значений экономических циклов по оценкам совокупного валового продукта (СВП). Предложенный подход к решению этой задачи базируется на интерпретации цикла в виде случайных колебаний функции доходов с некоторой собственной частотой, именуемых также узкополосным случайным процессом. Используемые при восстановлении траектории цикла операторы (преобразования Фурье, фильтрация и пр.) являются линейными, которым присуще свойство ассоциативности, позволяющее изменять их последовательность. Вследствие чего предложено начинать процедуру восстановления значений колебаний доходов с полосовой фильтрации функции СВП, а затем противодействовать эффекту инерционности оператора, формирующего оценки СВП. Учет особенностей узкополосного случайного процесса позволил создать упрощенную процедуру восстановления траектории цикла. На примере цикла Кузнеца показана ее приемлемость для задач практической эконометрики. Разработанный метод применим в задачах, требующих знания траектории рассматриваемого цикла.

Ключевые слова: экономический цикл, случайные колебания, траектория цикла, преобразования Фурье, амплитудная и фазовая частотные характеристики.

Классификация JEL: C02, C15, C22.

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